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Adaptive finite-time control for stochastic nonlinear systems using multi-dimensional Taylor network

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Abstract

In this paper, the problem of adaptive finite-time multi-dimensional Taylor network (MTN) control for a class of stochastic nonlinear systems is investigated. By combining the MTN-based approximate method and adaptive backstepping technique, a novel adaptive finite-time MTN control scheme is proposed. In this scheme, the MTNs are used to approximate the unknown nonlinear functions of the systems. The finite-time Lyapunov stability theory is utilized to prove the stability of the close-loop system. The proposed scheme can ensure that all signals in the closed-loop system are bounded in probability and the tracking error converges to a small neighborhood of the origin in a finite time. Three simulation examples are presented to show the effectiveness of the control scheme. It should be pointed that the adaptive MTN controller proposed in this paper has the advantages of fast computational speed and good real-time performance thanks to the simple structure of the MTN.

Keywords

Adaptive control, finite-time control, multi-dimensional Taylor network, stochastic nonlinear systems

Introduction

It is well known that stochastic disturbances exist widely in many practical systems and they are usually one of the factors of instability and complexity of the systems. Therefore, the controller design and performance analysis of stochastic systems are more difficult than that of general deterministic systems (Liu et al., 2016; Wang et al., 2014; Zhao et al., 2015). Driven by these problems, more and more attentions have been paid to the study of stochastic nonlinear systems (Deng and Krstic, 1997, 1999; Deng et al., 2001; Feng and Shi, 2017; Ji and Xi, 2006; Li et al., 2017; Pan and Basar, 1999; Wang et al., 2013; Wei et al., 2019; Wu et al., 2010; Yao et al., 2019; Zhong et al., 2015). So far, many control methods of deterministic systems are successfully extended to stochastic systems, such as adaptive control (Ji and Xi, 2006; Wei et al., 2019; Wu et al., 2010), fault tolerant control (Yao et al., 2019), approximation-based control (Li et al., 2017; Wang et al., 2013; Zhong et al., 2015), sliding mode control (Feng and Shi, 2017) and backstepping technique (Deng and Krstic, 1997, 1999; Deng et al., 2001; Pan and Basar, 1999). In particular, due to the existence of unknown functions and uncertainty of systems, the approximation-based adaptive control methods, such as neural network (NN)-based control (Hua et al., 2015; Li et al., 2009; Zhou et al., 2012) and fuzzy-based control (Li and Yue, 2015; Ma et al., 2019; Zhou et al., 2017), have become a research hotspot in recent years. For example, Hua et al. (2015) proposed an adaptive decentralized NN control scheme for a class of time-delay stochastic nonlinear systems. Ma et al. (2019) designed an adaptive fault-tolerant controller based on fuzzy logic systems for a class of nonstrict-feedback stochastic nonlinear systems. However, the above approximation-based control methods have their limitations (Han, 2018). For instance, NNs control methods have the disadvantages of uncertain structure and slow convergence speed. The design of fuzzy control is lack of systematicness and it is difficult to establish a systematic fuzzy control theory. Therefore, it is still a meaningful topic to construct a class of simple but effective adaptive control methods for stochastic nonlinear systems. Fortunately, the multi-dimensional Taylor network (MTN) has emerged, which can overcome the aforementioned shortcomings.

The MTN is a three-layer network structure including information input layer, middle layer composed of polynomials and output layer (Han, 2018). The MTN model has the advantages of strong learning ability and low computational cost due to simple structure (Han and Yan, 2020). Up to now, the model has been applied in many aspects such as nonlinear system identification (Zhou and Yan, 2014a, 2014b) and traffic flow prediction (Zhu et al., 2021). The MTN has been applied to the control problems of nonlinear

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systems due to its excellent approximation ability and many valuable results have been obtained in recent years (Han et al., 2021a, 2021b; Yan et al., 2018b; Zhu et al., 2020). Base on the above researches, the method was also generalized to the control problem of stochastic nonlinear systems (Han, 2020a, 2020b; Han and Yan, 2018, 2020; Han et al., 2018; Yan et al., 2018a). For example, for a class of stochastic nonlinear systems with input dead-zone, Han et al. (2018) proposed a MTN control method via backstepping technique. Han (2020a, 2020b) also studied the control problems of a class of stochastic nonlinear systems with input saturation constraints and input delay, respectively. Some novel MTN-based adaptive control strategies were proposed. However, although many meaningful achievements based on the MTN have been obtained, the MTN-based control method has not yet been applied to the field of finite-time control. One of the main goals of this paper is to extend this method to finite-time control of stochastic systems.

In fact, finite-time control problems are frequently encountered in engineering systems such as robot operating systems and aircraft control systems (Ding and Li, 2011). As a kind of time-optimal control, the main characteristic of finite-time control is that it can make the closed-loop system converges to equilibrium point in finite time and then remains there. Compared with the control systems in infinite time domain, the finite time control systems have better robust and anti-interference performance (Ding and Li, 2011; Hong et al., 2001). Therefore, finite-time control problems have attracted extensive attention during the past decades. For the first time, the finite-time Lyapunov stability theory was established to deal with the flutter problems caused by the sliding mode controller (Bhat and Bernstein,1998, 2000). On the base of the theory, the finite-time control problems of deterministic nonlinear systems have been studied by many authors (Hong, 2002; Kamalamiri et al., 2020; Wang et al., 2018; Yang et al., 2017). Subsequently, with the development of the study, the finite-time control for deterministic nonlinear systems has been generalized to stochastic nonlinear control systems and many significant achievements have been obtained (Chen and Jiao, 2010; Khoo et al., 2013; Liu and Zhu, 2020; Song and Zhai, 2017; Wang et al., 2019; Zhang et al., 2018). For example, Khoo et al. (2013) developed a systematic design algorithm for the stochastic nonlinear systems finite-time controller. Liu and Zhu (2020) studied the finite-time tracking control problems for uncertain pure feedback stochastic nonlinear systems with state constraints and established a novel adaptive finite-time neural network controller by utilizing backstepping technique. Wang et al. (2019) proposed a new adaptive finite-time fuzzy control strategy for a class of switched stochastic nonlinear systems. Zhang et al. (2018) constructed an adaptive finite-time controller for a class of stochastic nonlinear constraints systems with full state constraints. Although many adaptive finite-time fuzzy/NNs control methods for stochastic systems have been proposed, these methods have shortcoming of complex controller structure and poor real-time performance. Therefore, there is still much room for improvement of the existing research, which motivates our research.

Based on the above observations, this paper concentrates on the issue of adaptive finite-time control for a class of stochastic nonlinear systems. The MTNs are used to approximate the unknown functions. Based on the MTN-based approximate method and adaptive backstepping control technique, a new adaptive finite-time MTN control scheme is developed. Three simulation examples are presented to show the effectiveness of the provided control scheme. Compared with the existing works, the main contributions of this paper are as follows:

(1) In this paper, the MTN method is successfully used to deal with the finite-time control problems for stochastic nonlinear systems. A novel adaptive finite-time MTN controller is provided via the MTN-based approximate method and backstepping technique. In the current MTN-based research results, the control scheme based on the MTN for stochastic nonlinear systems (Han and Yan, 2018, 2020; Han et al., 2018) cannot be directly applied to the field of the finite-time control of stochastic systems.
(2) The unknown nonlinear functions of the systems are estimated with the help of approximate characteristic of the MTN. The computation burden of the control scheme is greatly alleviated thanks to the simple structure of the MTN. Therefore, the proposed MTN-based control scheme has a good real-time performance.

The rest of this paper is organized as follows. Section 2 presents some preliminaries, including the system descriptions, definitions and correlation theories. The main results of this paper are presented in Section 3. Three simulation examples are given in Sections 4 to show the effectiveness of the proposed scheme. Finally, the conclusions of the paper and suggestions for further research works are given in Section 5.

System description and preliminaries

The following notations are used throughout this paper. R denotes the set of all real numbers. For a given vector or matrix $\boldsymbol{x}$, $\mathrm{Tr}(\boldsymbol{x})$ denotes its trace when $\boldsymbol{x}$ is square matrix and $\|\boldsymbol{x}\|$ denotes the Euclidean norm for a vector $\boldsymbol{x}$. C^2 denotes the set of all functions with continuous 2-th partial derivative.

System descriptions

In this paper, the following stochastic nonlinear systems are considered

$$\begin{cases} dx_i = (x_{i+1} + f_i(\bar{\boldsymbol{x}}_i))dt + g_i^{\mathrm{T}}(\bar{\boldsymbol{x}}_i)d\boldsymbol{\omega} \\ \quad i = 1, 2, \ldots, n-1 \\ dx_n = (u + f_n(\bar{\boldsymbol{x}}_n))dt + g_n^{\mathrm{T}}(\bar{\boldsymbol{x}}_n)d\boldsymbol{\omega} \\ \quad y = x_1 \end{cases} \tag{1}$$

where $\bar{\boldsymbol{x}}_i = [x_1, x_2, \ldots, x_i]^{\mathrm{T}} \in R^i$ and $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in R^n$ are the state variables, $y \in R$ is the system output, $u \in R$ is the system input, $f_i(\cdot) : R^i \to R$ are the unknown continuous nonlinear functions with $f_i(\mathbf{0}) = 0$, $g_i(\cdot) : R^i \to R^{i \times r}$ are the unknown continuous functions with $g_i(\mathbf{0}) = \mathbf{0}$, and $\boldsymbol{\omega}$ is an independent r-dimensional standard Wiener process.

The objective of this paper is to design a control strategy for system (1) and achieve the following goals: (i) the output y of the system (1) can track the given reference signal y_d and tracking errors eventually converge to a small neighborhood of the origin in the finite-time; (ii) the systems (1) are semi-global finite-time stable in probability (SGFTSP).

In order to achieve the above objectives, the following assumption is introduced.

Assumption 1: The reference signal y_d and its up to n-th order derivatives are bounded and continuous.

Correlation theories

In order to introduce the related definitions and theorems of the stochastic nonlinear systems, we consider the stochastic nonlinear system in the following form

$$d\boldsymbol{x}(t) = f(\boldsymbol{x}(t))dt + g(\boldsymbol{x}(t))d\boldsymbol{\omega} \tag{2}$$

where $\boldsymbol{x} \in R^n$ is the system state, $f : R^n \to R^n$ and $g : R^n \to R^{n \times r}$ stand for unknown smooth nonlinear functions and satisfy $f(\boldsymbol{0}) = \boldsymbol{0}$ and $g(\boldsymbol{0}) = \boldsymbol{0}$. $\boldsymbol{\omega}$ is an independent r-dimensional standard Wiener process.

Definition 1: (Han and Yan, 2020; Han et al., 2018): Consider the stochastic nonlinear system (2), for any C^2 function $V(\boldsymbol{x}, t)$, the differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}V(\boldsymbol{x}, t) = \frac{\partial V}{\partial \boldsymbol{x}} f + \frac{1}{2}\mathrm{Tr}\left\{g^{\mathrm{T}} \frac{\partial^2 V}{\partial \boldsymbol{x}^2} g\right\} \tag{3}$$

Definition 2: (Wang et al., 2019): The solution x of the system (2) is SGFTSP if for all $x(t_0) = x_0$, there is $\varepsilon > 0$ and a setting time $T(\varepsilon, x_0) < \infty$ to make $\|x(t)\| < \varepsilon$, for all $t \geq t_0 + T$, where x_0 is the system initial state.

Lemma 1: (Wang et al., 2019): For $l_i \in R$, $i = 1, \ldots, n$, the following inequality holds

$$\left(\sum_{i=1}^{n} |l_i|\right)^{\gamma} \leq \sum_{i=1}^{n} |l_i|^{\gamma} \tag{4}$$

where $0 < \gamma \leq 1$.

Lemma 2: (Wang et al., 2019): For real variables α, β, we have the following inequality

$$|\alpha|^p |\beta|^q \leq \frac{p}{p+q} \varsigma |\alpha|^{p+q} + \frac{q}{p+q} \varsigma^{-\frac{p}{q}} |\beta|^{p+q} \tag{5}$$

where q, p, ς are any positive constants.

Lemma 3: (Wang et al., 2019): The trajectory of the system (2) is SGFTSP, if there exists a C^2 function $V(\boldsymbol{x}) : R^n \to R$ with $V(\boldsymbol{0}) = 0$, such that the following inequality holds

$$\mathcal{L}V(\boldsymbol{x}) \leq -aV^{\gamma}(\boldsymbol{x}) + \rho \tag{6}$$

In addition, $V(\boldsymbol{x})$ has the following feature

$$V^{\gamma}(\boldsymbol{x}) \leq \frac{\rho}{(1-\mu)a}, \quad \forall t \geq T_{\text{reach}}$$

where $a > 0$, $0 < \rho < \infty$, $0 < \mu < 1$ and T_{reach} is defined by

$$T_{\text{reach}} = \frac{1}{(1-\gamma)\mu a}\left\{V^{1-\gamma}(\boldsymbol{x}(0)) - \left(\frac{\rho}{(1-\mu)a}\right)^{\frac{1-\gamma}{\gamma}}\right\}$$

MTN

In this paper, the MTNs are used to estimate unknown smooth nonlinear functions of the systems (1). We recall the correlation theory of the MTN. See Han (2018) and Zhu et al. (2020) for details.

Lemma 4: (Yan et al., 2018a; Zhu et al., 2020): On a compact set Ω_Z, for $\forall \varepsilon > 0$, consider continuous nonlinear function $f(\boldsymbol{Z})$, there exists a MTN $\boldsymbol{\varphi}^{*\mathrm{T}} S_{m_n}(\boldsymbol{Z})$, such that

$$f(\boldsymbol{Z}) = \boldsymbol{\varphi}^{*\mathrm{T}} S_{m_n}(\boldsymbol{Z}) + \delta(\boldsymbol{Z}) \tag{7}$$

where $\boldsymbol{Z} = [z_1, z_1, \ldots, z_n]^{\mathrm{T}} \in \Omega_{\boldsymbol{Z}}$, and $\delta(\boldsymbol{Z})$ is the approximation error with $|\delta(\boldsymbol{Z})| \leq \varepsilon$, $\boldsymbol{\varphi}^* = [\varphi_1, \varphi_2, \ldots, \varphi_n]^{\mathrm{T}}$ is the optimal weight vector and defined by

$$\boldsymbol{\varphi}^* := \arg\min_{\boldsymbol{\varphi} \in R^N}\left\{\sup_{Z \in \Omega_Z} \left|f(\boldsymbol{Z}) - \boldsymbol{\varphi}^{\mathrm{T}} S(\boldsymbol{Z})\right|\right\}$$

Remark 1: Even though the MTN and NNs are all three-layer network structure in form, their principles are different. The NNs introduce basic function such as exponential functions, partial fractions and step to achieve nonlinear transformation, which have shortcomings of the long training time and high computational complexity. However, the MTN approximates the nonlinear functions by making full use of orthogonal polynomials. Compared with the NNs, the MTN has a simpler structure and better real-time performance (Han, 2018; Han and Yan, 2020; Han et al., 2021; Yan et al., 2018a; Zhou and Yan, 2014a, 2014b; Zhu et al., 2020; Zhu et al., 2021). Therefore, the MTN has better approximation capability in nonlinear system identification.

Main results

The process of designing adaptive MTN controller is divided into n steps. Firstly, the following change of coordinates is employed

$$\begin{aligned} z_1 &= x_1 - y_d \\ z_i &= x_i - \alpha_{i-1},\ i = 2, \ldots, n \end{aligned} \tag{8}$$

where α_{i-1} is a virtual control signal, which will be determined in $(i-1)$-th step.

Combining (1) and (8), one has

$$\begin{cases} dz_1 = [x_2 + f_1(\boldsymbol{x}_1) - \dot{y}_d]dt + g_1^{\mathrm{T}}(\boldsymbol{x}_1)d\boldsymbol{\omega} \\ dz_i = [x_{i+1} + f_i(\bar{\boldsymbol{x}}_i) - \Delta\alpha_{i-1}]dt + (g_i - \\ \quad \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} g_j)^{\mathrm{T}} d\boldsymbol{\omega} \quad 2 \leq i \leq n-1 \\ dz_n = [u + f_n(\bar{\boldsymbol{x}}_n) - \Delta\alpha_{n-1}]dt + (g_n - \\ \quad \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} g_j)^{\mathrm{T}} d\boldsymbol{\omega} \end{cases} \tag{9}$$

with $\Delta\alpha_{i-1} = \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}(f_j + x_{j+1}) + \sum_{j=0}^{i-1} \frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial\hat{\boldsymbol{\theta}}_j}\dot{\hat{\boldsymbol{\theta}}}_j + \frac{1}{2}\sum_{j,k=1}^{i-1} \frac{\partial^2\alpha_{i-1}}{\partial x_j \partial x_k} g_j^{\mathrm{T}} g_k$, where $\hat{\boldsymbol{\theta}}_j$ is the parameter vector of the MTN approximation and $\dot{\hat{\boldsymbol{\theta}}}_j$ is the adaptive law that will be defined later.

MTN-based backstepping design

Step 1: Consider the stochastic Lyapunov function candidate as follows

$$V_1 = \tfrac{1}{4}z_1^4 + \tfrac{1}{2}\tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1^{-1}\tilde{\boldsymbol{\theta}}_1 \tag{10}$$

where $\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_1^{\mathrm{T}} > 0$ is a symmetric positive definite matrix, $\tilde{\boldsymbol{\theta}}_1 = \boldsymbol{\theta}_1 - \hat{\boldsymbol{\theta}}_1$ is the parameter error vector.

According to Definition 1 and (10), we obtain

$$\mathcal{L}V_1 = z_1^3(x_2 + f_1 - \dot{y}_d) + \tfrac{3}{2}z_1^2 g_1^{\mathrm{T}} g_1 - \tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1^{-1}\dot{\hat{\boldsymbol{\theta}}}_1 \tag{11}$$

By using Young's inequality, we have

$$\frac{3}{2}z_1^2 g_1^{\mathrm{T}} g_1 \leq \frac{3}{4\xi_1^2}z_1^4\|g_1\|^4 + \frac{3}{4}\xi_1^2 \tag{12}$$

where ξ_1 is a design parameter.

Substituting (12) into (11), we can obtain

$$\mathcal{L}V_1 \leq z_1^3\left(x_2 + \bar{f}_1\right) - \frac{3}{2}z_1^4 + \frac{3}{4}\xi_1^2 - \tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1^{-1}\dot{\hat{\boldsymbol{\theta}}}_1 \tag{13}$$

where $\bar{f}_1 = f_1 - \dot{y}_d + \frac{3}{4\xi_1^2}z_1^4\|g_1\|^4 + \frac{3}{2}z_1$.

$\bar{f}_1$ is obviously an unknown function, and it cannot be directly used to construct virtual control signal α_1. According to Lemma 4, we can use a MTN to estimate $\bar{f}_1$. In other word, for $\forall\varepsilon_1 > 0$, there exists a MTN $\boldsymbol{\theta}_1^{\mathrm{T}} S_{m_1}(z_1)$, such that

$$\bar{f}_1 = \boldsymbol{\theta}_1^{\mathrm{T}} S_{m_1}(z_1) + \delta_1(z_1), |\delta_1(z_1)| \leq \varepsilon_1 \tag{14}$$

where $z_1 = [z_1]^{\mathrm{T}}$ is the input of MTN and $\delta_1(z_1)$ is the approximation error.

By using Young's inequality, we have

$$z_1^3 z_2 \leq \tfrac{3}{4}z_1^4 + \tfrac{1}{4}z_2^4 \tag{15}$$

$$z_1^3\delta_1 \leq \tfrac{3}{4}z_1^4 + \tfrac{1}{4}\varepsilon_1^4 \tag{16}$$

Substituting (8) and (14), (15) and (16) into (13) yields

$$\begin{aligned}\mathcal{L}V_1 \leq{}& z_1^3\left(\alpha_1 + \boldsymbol{\theta}_1^{\mathrm{T}} S_{m_1}(z_1)\right) + \frac{1}{4}\varepsilon_1^4 + \frac{1}{4}z_2^4 \\ &+ \frac{3}{4}\xi_1^2 - \tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1^{-1}\dot{\hat{\boldsymbol{\theta}}}_1\end{aligned} \tag{17}$$

Based on (17), taking the following intermediate virtual control signal α_1 as

$$\alpha_1 = -k_1 z_1^{4\gamma-3} - \hat{\boldsymbol{\theta}}_1^{\mathrm{T}} S_{m_1}(z_1) \tag{18}$$

where $k_1 > 0$ and $0 < \gamma < 1$ are design parameters.

Substituting (18) into (17), we have

$$\begin{aligned}\mathcal{L}V_1 \leq{}& -k_1 z_1^{4\gamma} + \frac{1}{4}z_2^4 + \frac{1}{4}\varepsilon_1^4 + \frac{3}{4}\xi_1^2 \\ &+ \tilde{\boldsymbol{\theta}}_1^{\mathrm{T}}\left(z_1^3 S_{m_1}(z_1) - \boldsymbol{\Gamma}_1^{-1}\dot{\hat{\boldsymbol{\theta}}}_1\right)\end{aligned} \tag{19}$$

Step 2: Consider the stochastic Lyapunov function candidate as follows

$$V_2 = V_1 + \tfrac{1}{4}z_2^4 + \tfrac{1}{2}\tilde{\boldsymbol{\theta}}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2^{-1}\tilde{\boldsymbol{\theta}}_2 \tag{20}$$

where $\boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}_2^{\mathrm{T}} > 0$ is a symmetric positive definite matrix, $\tilde{\boldsymbol{\theta}}_2 = \boldsymbol{\theta}_2 - \hat{\boldsymbol{\theta}}_2$ is the parameter error vector.

According to Definition 1 and (20), we can obtain

$$\begin{aligned}\mathcal{L}V_2 ={}& \mathcal{L}V_1 + z_2^3(x_3 + f_2 - \Delta\alpha_1) \\ &+ \frac{3}{2}z_2^2\tilde{g}_2^{\mathrm{T}}\tilde{g}_2 - \tilde{\boldsymbol{\theta}}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2^{-1}\dot{\hat{\boldsymbol{\theta}}}_2\end{aligned} \tag{21}$$

where $\tilde{g}_2 = g_2 - \sum_{j=1}^{1}\frac{\partial\alpha_1}{\partial x_j}g_j$.

By using Young's inequality, we have

$$\frac{3}{2}z_2^2\tilde{g}_2^{\mathrm{T}}\tilde{g}_2 \leq \frac{3}{4\xi_2^2}z_1^4\|\tilde{g}_2\|^4 + \frac{3}{4}\xi_2^2 \tag{22}$$

where ξ_2 is a design parameter.

Substituting (22) into (21), we have

$$\begin{aligned}\mathcal{L}V_2 \leq{}& \mathcal{L}V_1 + z_2^3\left(x_3 + \bar{f}_2\right) - \frac{7}{4}z_2^4 \\ &+ \frac{3}{4}\xi_2^2 - \tilde{\boldsymbol{\theta}}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2^{-1}\dot{\hat{\boldsymbol{\theta}}}_2\end{aligned} \tag{23}$$

where $\bar{f}_2 = f_2 - \sum_{j=1}^{1}\frac{\partial\alpha_1}{\partial x_j}(f_1 + x_2) + \sum_{j=0}^{1}\frac{\partial\alpha_1}{\partial y_d^{(j)}}y_d^{(j+1)} + \sum_{j=1}^{1}\frac{\partial\alpha_1}{\partial\hat{\boldsymbol{\theta}}_j}\dot{\hat{\boldsymbol{\theta}}}_j + \frac{1}{2}\sum_{j,k=1}^{1}\frac{\partial^2\alpha_1}{\partial x_j\partial x_k}g_j^{\mathrm{T}}g_k + \frac{3}{4\xi_2^2}z_2^4\|\tilde{g}_2\|^4 + \frac{7}{4}z_2$.

$\bar{f}_2$ is also an unknown function. When constructing virtual control signal α_2, a MTN can be used to approximate $\bar{f}_2$. Namely, for $\forall\varepsilon_2 > 0$, there exists a MTN $\boldsymbol{\theta}_2^{\mathrm{T}} S_{m_2}(z_2)$, such that

$$\bar{f}_2 = \boldsymbol{\theta}_2^{\mathrm{T}} S_{m_2}(z_2) + \delta_2(z_2), |\delta_2(z_2)| \leq \varepsilon_2 \tag{24}$$

where $z_2 = [z_1, z_2]^{\mathrm{T}}$ is the input of MTN and $\delta_2(z_2)$ is the approximation error.

By using Young's inequality, one has

$$z_2^3 z_3 \leq \tfrac{3}{4}z_2^4 + \tfrac{1}{4}z_3^4 \tag{25}$$

$$z_2^3\delta_2 \leq \tfrac{3}{4}z_2^4 + \tfrac{1}{4}\varepsilon_2^4 \tag{26}$$

Substituting (8), and (24), (25) and (26) into (23), we can get

$$\begin{aligned}\mathcal{L}V_2 \leq{}& \mathcal{L}V_1 + z_2^3\left(\alpha_2 + \boldsymbol{\theta}_2^{\mathrm{T}} S_{m_2}(z_2)\right) + \frac{1}{4}\varepsilon_2^4 + \frac{1}{4}z_3^4 \\ &- \frac{1}{4}z_2^4 + \frac{3}{4}\xi_2^2 - \tilde{\boldsymbol{\theta}}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2^{-1}\dot{\hat{\boldsymbol{\theta}}}_2\end{aligned} \tag{27}$$

According to (27), designing the following intermediate virtual control signal α_2 as

$$\alpha_2 = -k_2 z_2^{4\gamma-3} - \hat{\boldsymbol{\theta}}_2^{\mathrm{T}} S_{m_2}(\boldsymbol{z}_2) \tag{28}$$

where $k_2 > 0$ and $0 < \gamma < 1$ are design parameters.

Substituting (19) and (28) into (27), we have

$$\begin{aligned}\mathcal{L}V_2 \leq & \sum_{j=1}^{2} -k_j z_j^{4\gamma} + \frac{1}{4} z_3^4 + \frac{1}{4}\sum_{j=1}^{2} \varepsilon_j^4 + \frac{3}{4}\sum_{j=1}^{2} \xi_j^2 \\ & + \sum_{j=1}^{2} \tilde{\boldsymbol{\theta}}_j^{\mathrm{T}} \left(z_j^3 S_{m_j}(\boldsymbol{z}_j) - \boldsymbol{\Gamma}_j^{-1} \dot{\hat{\boldsymbol{\theta}}}_j \right)\end{aligned} \tag{29}$$

Step $i (i = 3, \ldots, n-1)$: Consider the stochastic Lyapunov function candidate as follows

$$V_i = V_{i-1} + \tfrac{1}{4} z_i^4 + \tfrac{1}{2} \tilde{\boldsymbol{\theta}}_i^{\mathrm{T}} \boldsymbol{\Gamma}_i^{-1} \tilde{\boldsymbol{\theta}}_i \tag{30}$$

where $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_i^{\mathrm{T}} > 0$ is a symmetric positive definite matrix, $\tilde{\boldsymbol{\theta}}_i = \boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i$ is the parameter error vector.

According to Definition 1 and (30), we can obtain that

$$\begin{aligned}\mathcal{L}V_i = & \mathcal{L}V_{i-1} + z_i^3 (x_{i+1} + f_i - \Delta\alpha_{i-1}) \\ & - \tilde{\boldsymbol{\theta}}_i^{\mathrm{T}} \boldsymbol{\Gamma}_i^{-1} \dot{\hat{\boldsymbol{\theta}}}_i + \frac{3}{2} z_i^2 \tilde{g}_i^{\mathrm{T}} \tilde{g}_i\end{aligned} \tag{31}$$

where $\tilde{g}_i = g_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} g_j$.

By using Young's inequality, we have

$$\frac{3}{2} z_i^2 \tilde{g}_i^{\mathrm{T}} \tilde{g}_i \leq \frac{3}{4\xi_i^2} z_i^4 \|\tilde{g}_i\|^4 + \frac{3}{4} \xi_i^2 \tag{32}$$

where ξ_i is a design parameter.

Substituting (32) into (31) yields

$$\begin{aligned}\mathcal{L}V_i \leq & \mathcal{L}V_{i-1} + z_i^3 \left(x_{i+1} + \bar{f}_i \right) - \frac{7}{4} z_i^4 \\ & + \frac{3}{4} \xi_i^2 - \tilde{\boldsymbol{\theta}}_i^{\mathrm{T}} \boldsymbol{\Gamma}_i^{-1} \dot{\hat{\boldsymbol{\theta}}}_i\end{aligned} \tag{33}$$

where $\bar{f}_i = f_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} (f_j + x_{j+1}) + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \boldsymbol{\theta}_j} \dot{\hat{\boldsymbol{\theta}}}_j + \frac{1}{2} \sum_{j,k=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_j \partial x_k} g_j^{\mathrm{T}} g_k + \frac{3}{4\xi_i^2} z_i^4 \|\tilde{g}_i\|^4 + \frac{7}{4} z_i$.

Similar to functions $\bar{f}_1$ and $\bar{f}_2$, $\bar{f}_i$ can be approximated by using a MTN in the design of virtual control signal α_i. Namely, for $\forall \varepsilon_i > 0$, there exists a MTN $\boldsymbol{\theta}_i^{\mathrm{T}} S_{m_i}(\boldsymbol{z}_i)$, such that

$$\bar{f}_i = \boldsymbol{\theta}_i^{\mathrm{T}} S_{m_i}(\boldsymbol{z}_i) + \delta_i(\boldsymbol{z}_i), |\delta_i(\boldsymbol{z}_i)| \leq \varepsilon_i \tag{34}$$

where $\boldsymbol{z}_i = [z_1, \cdots, z_i]^{\mathrm{T}}$ is the input of MTN and $\delta_i(\boldsymbol{z}_i)$ is the approximation error.

By using Young's inequality, one has

$$z_i^3 z_{i+1} \leq \tfrac{3}{4} z_i^4 + \tfrac{1}{4} z_{i+1}^4 \tag{35}$$

$$z_i^3 \delta_i \leq \tfrac{3}{4} z_i^4 + \tfrac{1}{4} \varepsilon_i^4 \tag{36}$$

Substituting (8), and (34), (35) and (36) into (33) yields

$$\begin{aligned}\mathcal{L}V_i \leq & \mathcal{L}V_{i-1} + z_i^3 \left(\alpha_i + \boldsymbol{\theta}_i^{\mathrm{T}} S_{m_i}(\boldsymbol{z}_i) \right) + \frac{1}{4} \varepsilon_i^4 \\ & + \frac{1}{4} z_{i+1}^4 - \frac{1}{4} z_i^4 + \frac{3}{4} \xi_i^2 - \tilde{\boldsymbol{\theta}}_i^{\mathrm{T}} \boldsymbol{\Gamma}_i^{-1} \dot{\hat{\boldsymbol{\theta}}}_i\end{aligned} \tag{37}$$

Based on (37), designing the following intermediate virtual control signal α_i as

$$\alpha_i = -k_i z_i^{4\gamma-3} - \hat{\boldsymbol{\theta}}_i^{\mathrm{T}} S_{m_i}(\boldsymbol{z}_i) \tag{38}$$

where $k_i > 0$ and $0 < \gamma < 1$ are design parameters.

Substituting (29) and (38) into (37), we have

$$\begin{aligned}\mathcal{L}V_i \leq & \sum_{j=1}^{i} -k_j z_j^{4\gamma} + \frac{1}{4} z_{i+1}^4 + \frac{1}{4}\sum_{j=1}^{i} \varepsilon_j^4 + \frac{3}{4}\sum_{j=1}^{i} \xi_j^2 \\ & + \sum_{j=1}^{i} \tilde{\boldsymbol{\theta}}_j^{\mathrm{T}} \left(z_j^3 S_{m_j}(\boldsymbol{z}_j) - \boldsymbol{\Gamma}_j^{-1} \dot{\hat{\boldsymbol{\theta}}}_j \right)\end{aligned} \tag{39}$$

Step n: Consider the stochastic Lyapunov function candidate as follows

$$V_n = V_{n-1} + \tfrac{1}{4} z_n^4 + \tfrac{1}{2} \tilde{\boldsymbol{\theta}}_n^{\mathrm{T}} \boldsymbol{\Gamma}_n^{-1} \tilde{\boldsymbol{\theta}}_n \tag{40}$$

where $\boldsymbol{\Gamma}_n = \boldsymbol{\Gamma}_n^{\mathrm{T}} > 0$ is a symmetric positive definite matrix, $\tilde{\boldsymbol{\theta}}_n = \boldsymbol{\theta}_n - \hat{\boldsymbol{\theta}}_n$ is the parameter error vector.

According to Definition 1 and (40), we can obtain

$$\begin{aligned}\mathcal{L}V_n = & \mathcal{L}V_{n-1} + z_n^3 (u + f_n - \Delta\alpha_{n-1}) \\ & - \tilde{\boldsymbol{\theta}}_n^{\mathrm{T}} \boldsymbol{\Gamma}_n^{-1} \dot{\hat{\boldsymbol{\theta}}}_n + \frac{3}{2} z_n^2 \tilde{g}_n^{\mathrm{T}} \tilde{g}_n\end{aligned} \tag{41}$$

where $\tilde{g}_n = g_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} g_j$.

By using Young's inequality, we have

$$\frac{3}{2} z_n^2 \tilde{g}_n^{\mathrm{T}} \tilde{g}_n \leq \frac{3}{4\xi_n^2} z_n^4 \|\tilde{g}_n\|^4 + \frac{3}{4} \xi_n^2 \tag{42}$$

where ξ_n is a design parameter.

Substituting (42) into (41), we have

$$\begin{aligned}\mathcal{L}V_n \leq & \mathcal{L}V_{n-1} + z_n^3 \left(u + \bar{f}_n \right) - \frac{7}{4} z_n^4 \\ & + \frac{3}{4} \xi_n^2 - \tilde{\boldsymbol{\theta}}_n^{\mathrm{T}} \boldsymbol{\Gamma}_n^{-1} \dot{\hat{\boldsymbol{\theta}}}_n\end{aligned} \tag{43}$$

where $\bar{f}_n = f_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} (f_j + x_{j+1}) + \sum_{j=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \boldsymbol{\theta}_j} \dot{\hat{\boldsymbol{\theta}}}_j + \frac{1}{2} \sum_{j,k=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_j \partial x_k} g_j^{\mathrm{T}} g_k + \frac{3}{4\xi_n^2} z_n^4 \|\tilde{g}_n\|^4 + z_n$.

Clearly, $\bar{f}_n$ also needs to be estimated by using a MTN when constructing actual control signal u. In other word, for $\forall \varepsilon_n > 0$, there exists a MTN $\boldsymbol{\theta}_n^{\mathrm{T}} S_{m_n}(\boldsymbol{z}_n)$, such that

$$\bar{f}_n = \boldsymbol{\theta}_n^{\mathrm{T}} S_{m_n}(\boldsymbol{z}_n) + \delta_n(\boldsymbol{z}_n), |\delta_n(\boldsymbol{z}_n)| \leq \varepsilon_n \tag{44}$$

where $\boldsymbol{z}_n = [z_1, \cdots, z_n]^{\mathrm{T}}$ is the input of MTN and $\delta_n(\boldsymbol{z}_n)$ is the approximation error.

By using Young's inequality, we have

$$z_n^3\delta_n \leqslant \tfrac{3}{4}z_n^4 + \tfrac{1}{4}\varepsilon_n^4 \tag{45}$$

Substituting (8), (44) and (45) into (43), we can obtain that

$$\begin{aligned}\mathcal{L}V_n \leqslant{}& \mathcal{L}V_{n-1} + z_n^3\left(u + \boldsymbol{\theta}_n^{\mathrm{T}}S_{m_n}(z_n)\right) + \frac{1}{4}\varepsilon_n^4\\ &-\frac{1}{4}z_n^4 + \frac{3}{4}\xi_n^2 - \tilde{\boldsymbol{\theta}}_n^{\mathrm{T}}\boldsymbol{\Gamma}_n^{-1}\dot{\hat{\boldsymbol{\theta}}}_n\end{aligned} \tag{46}$$

According to (46), taking the following control input u as

$$u = -k_n z_n^{4\gamma-3} - \hat{\boldsymbol{\theta}}_n^{\mathrm{T}}S_{m_n}(z_n) \tag{47}$$

where $k_n > 0$ and $0 < \gamma < 1$ are design parameters.

Substituting (39) and (47) into (46) results in

$$\begin{aligned}\mathcal{L}V_n \leqslant{}& \sum_{j=1}^{n} -k_j z_j^{4\gamma} + \frac{1}{4}\sum_{j=1}^{n}\varepsilon_j^4 + \frac{3}{4}\sum_{j=1}^{n}\xi_j^2\\ &+ \sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\left(z_j^3 S_{m_j}(z_j) - \boldsymbol{\Gamma}_j^{-1}\dot{\hat{\boldsymbol{\theta}}}_j\right)\end{aligned} \tag{48}$$

Based on the foregoing analysis and (48), the adaptive laws $\dot{\hat{\boldsymbol{\theta}}}_j$ can be designed as

$$\dot{\hat{\boldsymbol{\theta}}}_j = \boldsymbol{\Gamma}_j S_{m_j}(z_j)z_j^3 - \boldsymbol{\Gamma}_j\eta_j\hat{\boldsymbol{\theta}}_j,\ 1 \leqslant j \leqslant n \tag{49}$$

Substituting (49) into (48) yields

$$\begin{aligned}\mathcal{L}V_n \leqslant{}& \sum_{j=1}^{n} -k_j z_j^{4\gamma} + \frac{1}{4}\sum_{j=1}^{n}\varepsilon_j^4 + \frac{1}{4}\varepsilon^4\\ &+ \frac{3}{4}\sum_{j=1}^{n}\xi_j^2 + \sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\eta_j\hat{\boldsymbol{\theta}}_j\end{aligned} \tag{50}$$

According to the definition of $\hat{\boldsymbol{\theta}}_j$, we have

$$\eta_j\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\hat{\boldsymbol{\theta}}_j = \eta_j\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}(\boldsymbol{\theta}_j - \tilde{\boldsymbol{\theta}}_j) \leqslant -\tfrac{\eta_j}{2}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_j + \tfrac{\eta_j}{2}\left\|\boldsymbol{\theta}_j\right\|^2 \tag{51}$$

Substituting (51) into (50), we have

$$\begin{aligned}\mathcal{L}V_n \leqslant{}& \sum_{j=1}^{n} -k_j z_j^{4\gamma} + \frac{1}{4}\sum_{j=1}^{n}\varepsilon_j^4 + \frac{3}{4}\sum_{j=1}^{n}\xi_j^2\\ &+ \sum_{j=1}^{n}\left(-\frac{\eta_j}{2}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_j + \frac{\eta_j}{2}\left\|\boldsymbol{\theta}_j\right\|^2\right)\end{aligned} \tag{52}$$

Then, subtracting and adding the term $\left(\frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\boldsymbol{\Gamma}_j^{-1}\tilde{\boldsymbol{\theta}}_j\right)^{\gamma}$ in (52), we can get

$$\begin{aligned}\mathcal{L}V_n \leqslant{}& \sum_{j=1}^{n} -k_j z_j^{4\gamma} - \left(\frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\boldsymbol{\Gamma}_j^{-1}\tilde{\boldsymbol{\theta}}_j\right)^{\gamma}\\ &+ \frac{3}{4}\sum_{j=1}^{n}\xi_j^2 + \sum_{j=1}^{n}\frac{\eta_j}{2}\left\|\boldsymbol{\theta}_j\right\|^2 - \sum_{j=1}^{n}\frac{\eta_j}{2}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_j\\ &+ \left(\frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\boldsymbol{\Gamma}_j^{-1}\tilde{\boldsymbol{\theta}}_j\right)^{\gamma} + \frac{1}{4}\sum_{j=1}^{n}\varepsilon_j^4\end{aligned} \tag{53}$$

Applying Lemma 2, we have

$$\begin{aligned}\left(\frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\boldsymbol{\Gamma}_j^{-1}\tilde{\boldsymbol{\theta}}_j\right)^{\gamma} \leqslant{}& (1-\gamma)e^{\frac{\gamma\ln\gamma}{1-\gamma}}\\ &+ \frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\boldsymbol{\Gamma}_j^{-1}\tilde{\boldsymbol{\theta}}_j\end{aligned} \tag{54}$$

Substituting (54) into (53) yields

$$\begin{aligned}\mathcal{L}V_n \leqslant{}& \sum_{j=1}^{n} -k_j z_j^{4\gamma} - \left(\frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\boldsymbol{\Gamma}_j^{-1}\tilde{\boldsymbol{\theta}}_j\right)^{\gamma}\\ &+ \frac{3}{4}\sum_{j=1}^{n}\xi_j^2 + \sum_{j=1}^{n}\frac{\eta_j}{2}\left\|\boldsymbol{\theta}_j\right\|^2 + (1-\gamma)e^{\frac{\gamma\ln\gamma}{1-\gamma}}\\ &- \sum_{j=1}^{n}\frac{\eta_j}{2}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_j + \frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\boldsymbol{\Gamma}_j^{-1}\tilde{\boldsymbol{\theta}}_j + \frac{1}{4}\sum_{j=1}^{n}\varepsilon_j^4\end{aligned} \tag{55}$$

According to (55), we can obtain

$$\begin{aligned}\mathcal{L}V_n \leqslant{}& \sum_{j=1}^{n} -k_j z_j^{4\gamma} - \left(\frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\boldsymbol{\Gamma}_j^{-1}\tilde{\boldsymbol{\theta}}_j\right)^{\gamma}\\ &+ \frac{1}{4}\sum_{j=1}^{n}\varepsilon_j^4 + \frac{3}{4}\sum_{j=1}^{n}\xi_j^2 + \sum_{j=1}^{n}\frac{\eta_j}{2}\left\|\boldsymbol{\theta}_j\right\|^2\\ &+ (1-\gamma)e^{\frac{\gamma\ln\gamma}{1-\gamma}}\end{aligned} \tag{56}$$

From Lemma 1, we have

$$-\sum_{j=1}^{n}k_j z_j^{4\gamma} \leqslant -k\left(\sum_{j=1}^{n}z_j^4\right)^{\gamma} \tag{57}$$

where $k = \min\{k_j | j = 1, \ldots, n\}$.

Substituting (57) into (56) yields

$$\begin{aligned}\mathcal{L}V_n \leqslant{}& -k\left(\sum_{j=1}^{n}z_j^4\right)^{\gamma} - \left(\frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{\theta}}_j^{\mathrm{T}}\boldsymbol{\Gamma}_j^{-1}\tilde{\boldsymbol{\theta}}_j\right)^{\gamma}\\ &+ \frac{1}{4}\sum_{j=1}^{n}\varepsilon_j^4 + \frac{3}{4}\sum_{j=1}^{n}\xi_j^2 + \sum_{j=1}^{n}\frac{\eta_j}{2}\left\|\boldsymbol{\theta}_j\right\|^2\\ &+ (1-\gamma)e^{\frac{\gamma\ln\gamma}{1-\gamma}}\end{aligned}$$

Let $a = \min\{4k, 2^{\gamma}\}$ and $\rho = \frac{1}{4}\sum_{j=1}^{n}\varepsilon_j^4 + \frac{3}{4}\sum_{j=1}^{n}\xi_j^2 + \sum_{j=1}^{n}\frac{\eta_j}{2}\left\|\boldsymbol{\theta}_j\right\|^2 + (1-\gamma)e^{\frac{\gamma\ln\gamma}{1-\gamma}}$, we have

$$\mathcal{L}V_n \leqslant -aV_n^{\gamma} + \rho \tag{58}$$

Result analysis

Based on the above design scheme and analysis process, we can summarize the main results of this paper as the following theorem.

Theorem 1: Under Assumption 1, for the stochastic nonlinear systems (1), the intermediate control signals (18), (28) and

(38), the control input (47) and the adaptive laws (49) can ensure that all the signals in the closed-loop systems are bounded in probability, and the tracking error remains in a small neighborhood of the origin in a finite time.

Proof: Consider the following stochastic Lyapunov function for the stochastic nonlinear systems (1)

$$V = V_n = \frac{1}{4}\sum_{i=1}^{n} z_i^4 + \frac{1}{2}\sum_{i=1}^{n} \tilde{\boldsymbol{\theta}}_i^{\mathrm{T}}\boldsymbol{\Gamma}_i^{-1}\tilde{\boldsymbol{\theta}}_i \tag{59}$$

According to Lemma 3, (58) and (59), we have

$$T_{\text{reach}} = \frac{1}{(1-\gamma)\mu a}\left\{V^{1-\gamma}(x(0)) - \left(\frac{\rho}{(1-\mu)a}\right)^{\frac{1-\gamma}{\gamma}}\right\}$$

with $0<\mu<1$.

Then, it follows from Lemma 3 that for $\forall t > T_{\text{reach}}$, $V^{\gamma}(x(0)) < \frac{\rho}{(1-\mu)a}$, which implies that any signal of the closed-loop systems (1) is SGFTSP.

In addition, from (59), one has

$$|y - y_d| \leqslant 2\left(\frac{\rho}{(1-\mu)a}\right)^{\frac{1}{4\gamma}}, \quad \forall t > T_{\text{reach}} \tag{60}$$

This theorem completes the proof.

Simulation

In this section, three simulation examples are given to show the effectiveness of the proposed control scheme in this paper.

Example 1 (Numerical example): Consider the third-order stochastic nonlinear system

$$\begin{cases} dx_1 = (x_2 - 0.2x_1 \sin(x_1))dt + x_1 d\boldsymbol{\omega} \\ dx_2 = \left(x_3 - x_1 \cos(x_2^2)\right)dt + \sin(x_1)x_2 d\boldsymbol{\omega} \\ dx_3 = \left(u + x_2 x_3^2\right)dt + \sin(x_3)d\boldsymbol{\omega} \\ \quad y = x_1 \end{cases} \tag{61}$$

with the initial state $[x_1(0), x_2(0), x_3(0)]^{\mathrm{T}} = [0,0,0]^{\mathrm{T}}$.

Consider the reference signal $y_d = 0.5\sin(t)$. Based on the above controller design process, the actual control law $u = -k_3 z_3^{4\gamma-3} - \hat{\boldsymbol{\theta}}_3^{\mathrm{T}} S_{m_3}(z_3)$, the intermediate control signals $\alpha_i = -k_i z_i^{4\gamma-3} - \hat{\boldsymbol{\theta}}_i^{\mathrm{T}} S_{m_i}(z_i), 1 \leqslant i \leqslant 2$ and the adaptive laws $\dot{\hat{\boldsymbol{\theta}}}_i = \boldsymbol{\Gamma}_i S_{m_i}(z_i) z_i^3 - \boldsymbol{\Gamma}_i \eta_i \hat{\boldsymbol{\theta}}_i$, $1 \leqslant i \leqslant 3$, where $z_1 = [z_1]^{\mathrm{T}}$, $z_2 = [z_1, z_2]^{\mathrm{T}}$, $z_3 = [z_1, z_2, z_3]^{\mathrm{T}}$, $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$ and $z_3 = x_3 - \alpha_2$. Moreover, the following design parameters are applied to the simulation: $\eta_1 = 4$, $\eta_2 = 10$, $\eta_3 = 0.6$, $k_1 = 7.5$, $k_2 = 20$, $k_3 = 20$, $\boldsymbol{\Gamma}_1 = 0.2\boldsymbol{I}_5$, $\boldsymbol{\Gamma}_2 = 10\boldsymbol{I}_9$, $\boldsymbol{\Gamma}_3 = 0.1\boldsymbol{I}_9$, and $\gamma = 0.99$.

The simulation results of Example 1 are given in Figures 1–4, respectively. Figure 1 illustrates the trajectories of the reference signal y_d and the system output y. Figure 2 displays the response of control signal u. Figure 3 shows that the state variables x_2 and x_3 are bounded. Figure 4 shows that the tracking error $y - y_d$ converges to a small neighborhood around the origin in the finite-time.

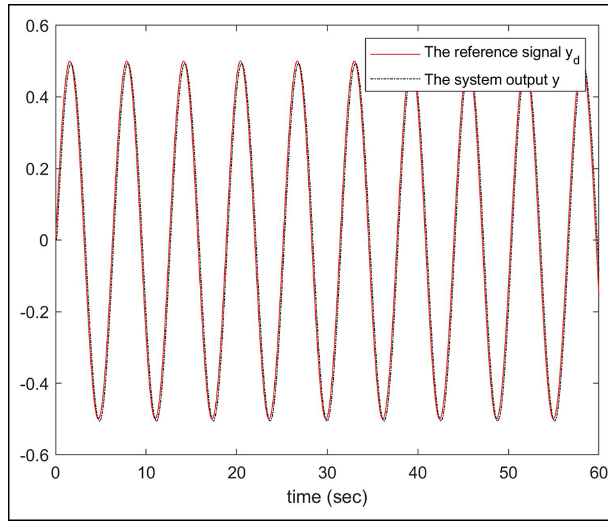

Figure 1. The trajectories of the system output y and the reference signal y_d of Example 1.

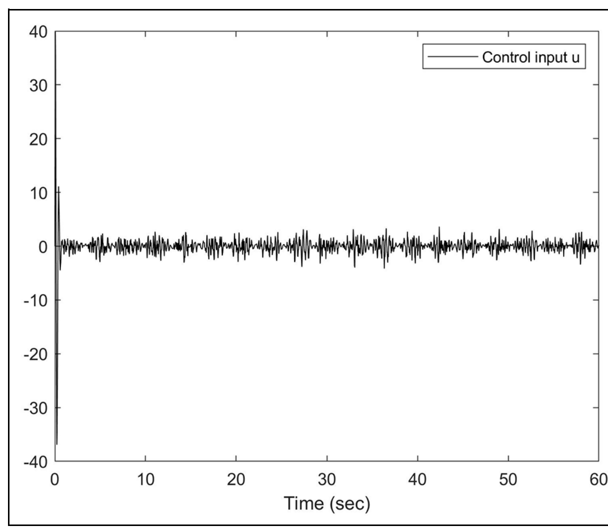

Figure 2. The trajectory of the system control input u of Example 1.

Example 2 (Practical example): Considering the one-link manipulator system with the influence of stochastic disturbance (Wang et al., 2015). The system can be described as

$$\begin{cases} dx_1 = x_2 dt + x_1 d\boldsymbol{\omega} \\ dx_2 = (x_3 - 2\cos(x_1) - x_2)dt + \sin(x_1)x_2 d\boldsymbol{\omega} \\ dx_3 = (u - 2x_2 - x_3)dt \\ y = x_1 \end{cases} \tag{62}$$

with the initial state $[x_1(0), x_2(0), x_3(0)]^{\mathrm{T}} = [0,0,0]^{\mathrm{T}}$.

Consider the reference signal $y_d = 0.8\sin(t)$. Based on the above controller design process, the actual control law u, the intermediate control signals α_i and the adaptive laws $\dot{\hat{\theta}}_i$ are respectively chosen as

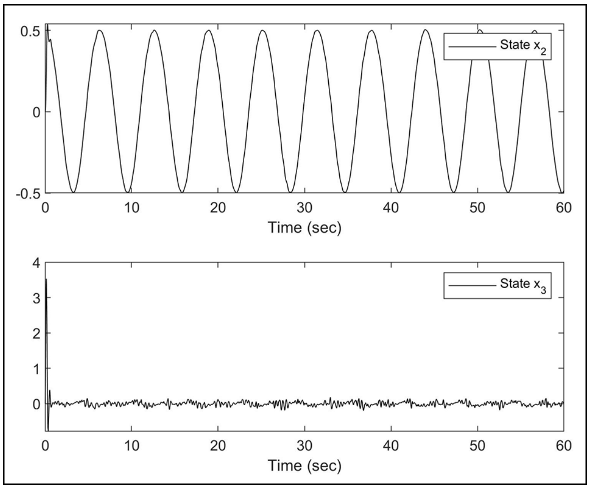

Figure 3. The trajectories of the system state variables x_2 and x_3 of Example 1.

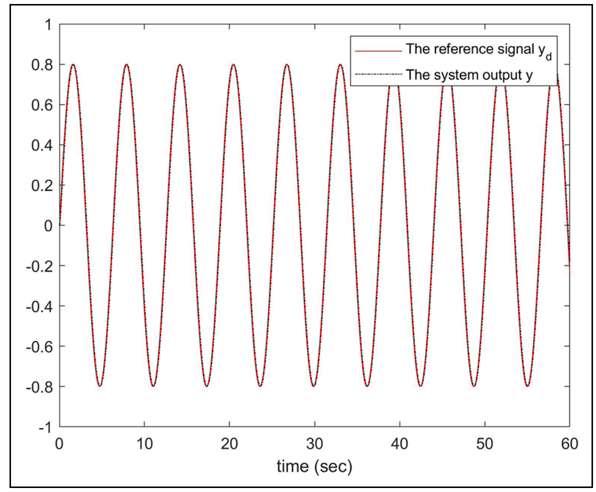

Figure 5. The trajectories of the system output and the reference signal y_d of Example 2.

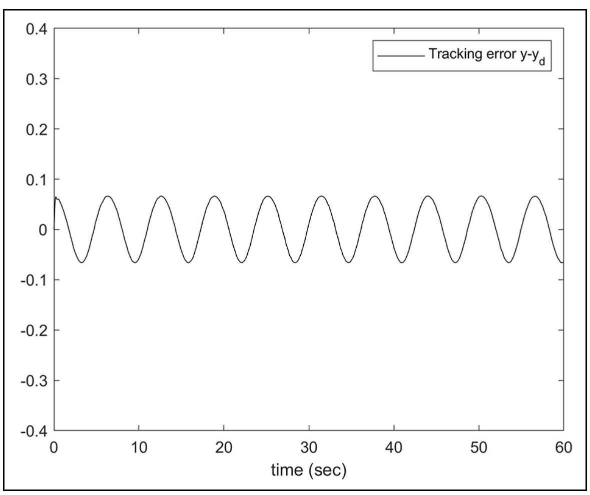

Figure 4. The trajectory of the tracking error $y - y_d$ of Example 1.

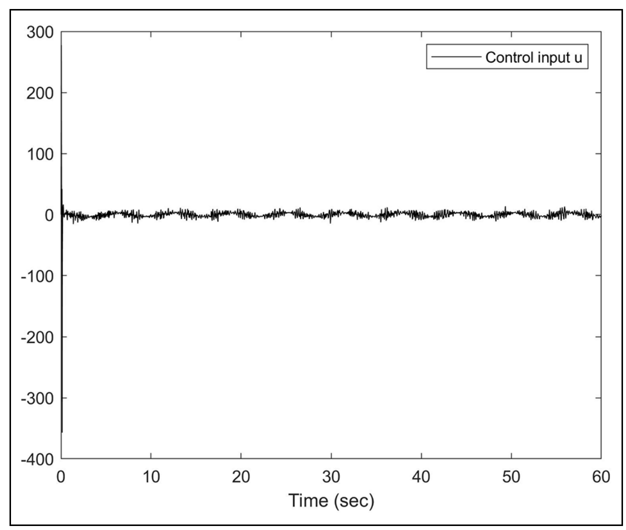

Figure 6. The trajectory of the system control input u of Example 2.

$$u = -k_3 z_3^{4\gamma-3} - \hat{\boldsymbol{\theta}}_3^{\mathrm{T}} S_{m_3}(z_3)$$

$$\alpha_i = -k_i z_i^{4\gamma-3} - \hat{\boldsymbol{\theta}}_i^{\mathrm{T}} S_{m_i}(z_i), 1 \leqslant i \leqslant 2$$

$$\dot{\hat{\boldsymbol{\theta}}}_i = \boldsymbol{\Gamma}_i S_{m_i}(z_i) z_i^3 - \boldsymbol{\Gamma}_i \eta_i \hat{\boldsymbol{\theta}}_i, \ 1 \leqslant i \leqslant 3$$

where $z_1 = [z_1]^{\mathrm{T}}$, $z_2 = [z_1, z_2]^{\mathrm{T}}$, $z_3 = [z_1, z_2, z_3]^{\mathrm{T}}$, $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$ and $z_3 = x_3 - \alpha_2$. The design parameters are chosen as : $\eta_1 = 2$, $\eta_2 = 10$, $\eta_3 = 0.5$, $k_1 = 12$, $k_2 = 32$, $k_3 = 26$, $\boldsymbol{\Gamma}_1 = 0.4\boldsymbol{I}_5$, $\boldsymbol{\Gamma}_2 = 8\boldsymbol{I}_9$, $\boldsymbol{\Gamma}_3 = 0.4\boldsymbol{I}_9$, and $\gamma = 0.99$.

The simulation results of Example 2 are given in Figures 5–8, respectively. The results show that the system output y can accurately track the reference signal y_d, and the desired tracking performance is achieved in finite time, which further verifies the effectiveness of the proposed control scheme in this paper.

Remark 2: Both numerical and practical examples show that all signals in the closed-loop systems are bounded and the tracking errors can converge to a small neighborhood of zero in a finite time. The good tracking performance can be achieved by choosing the proper design parameters η_i, k_i, $\boldsymbol{\Gamma}_i$.

Example 3 (Comparative example): To further demonstrate the superiority of the proposed control scheme, a comparative experiment between the adaptive finite-time MTN and RBF neural network (RBFNN) control methods was carried out on the basis of Example 1. The specific way is to replace the MTNs with RBFNNs in the control strategy proposed in Theorem 1, respectively. The simulation comparison results are illustrated in Figure 9.

As we see from Figure 9, although both MTN-based controller and RBFNN-based controller can realize the tracking control, the former possesses lower computational complexity and more satisfactory performance than the latter, as shown

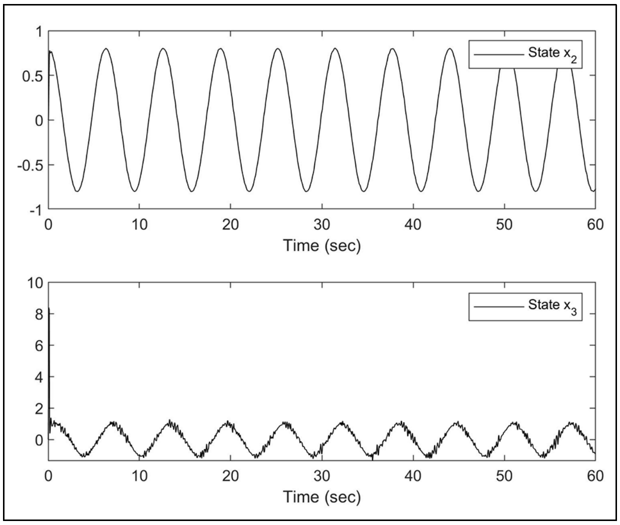

Figure 7. The trajectories of the system state variables x_2 and x_3 of Example 2.

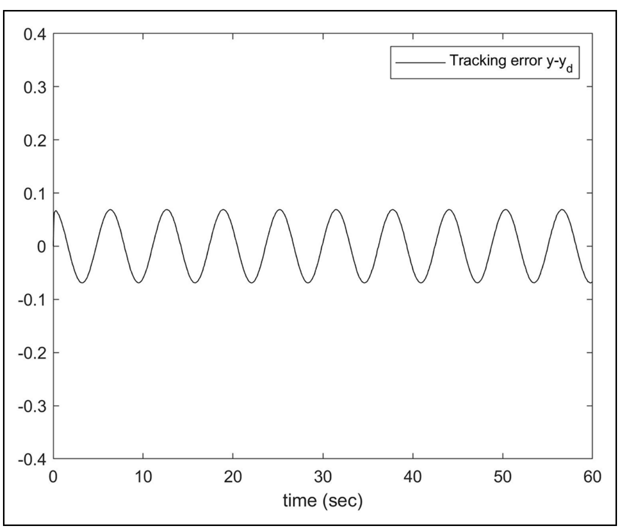

Figure 8. The trajectory of the tracking error $y - y_d$ of Example 2.

in the local enlarged diagram of Figure 9. The comparative experiment results further verify the effectiveness of the proposed control scheme.

Conclusion

In this paper, a new adaptive finite-time MTN control scheme is developed to solve the control problems of stochastic nonlinear systems. With the help of the MTN method and backstepping technique, an adaptive finite-time MTN control scheme is proposed. In the controller design process, the MTNs are used to handle the unknown functions, which greatly reduce the computation complexity of the proposed scheme. Based on finite-time Lyapunov stability theory, the

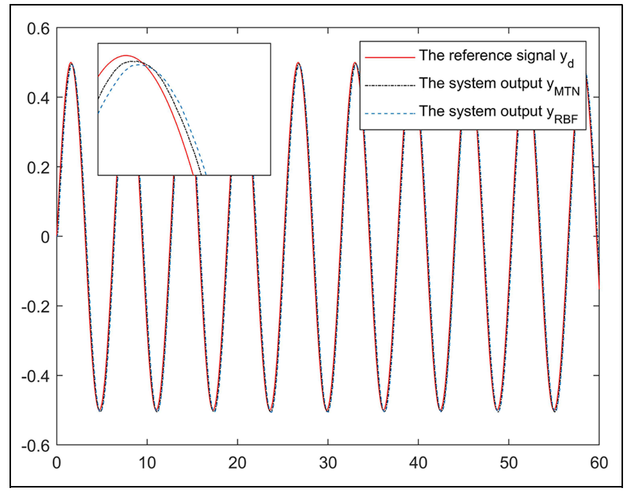

Figure 9. The tracking performances of Example 1 under two cases.

stability of the closed-loop systems can be proved. The proposed scheme can ensure that the closed-loop systems are SGFTSP, and tracking errors converge to a small neighborhood of zero in a finite time. Finally, the simulation results of three examples demonstrate the effectiveness of the proposed control scheme in this paper. In our future research, we could further focus on the adaptive finite-time MTN control problems for the other stochastic nonlinear systems, such as state constraints systems and input delay systems.

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References

Bhat SP and Bernstein DS (1998) Continuous finite-time stabilization of the translational and rotational double integrators. *IEEE Transactions on Automatic Control* 43(5): 678–682.

Bhat SP and Bernstein DS (2000) Finite-time stability of continuous autonomous systems. *SIAM Journal on Control and Optimization* 38(3): 751–766.

Chen WS and Jiao LC (2010) Finite-time stability theorem of stochastic nonlinear systems. *Automatica* 46(12): 2105–2108.

Deng H and Krstic M (1997) Stochastic nonlinear stabilization- I: A backstepping design. *Systems & Control Letters* 32(3): 143–150.

Deng H and Krstic M (1999) Output-feedback stochastic nonlinear stabilization. *IEEE Transactions on Automatic Control* 44(2): 328–333.

Deng H, Krstic M and Williams RJ (2001) Stabilization of stochastic nonlinear systems driven by noise of unknown covariance. *IEEE Transactions on Automatic Control* 46(8): 1237–1253.

Ding SH and Li SH (2011) A survey for finite-time control problems. *Control and Decision* 26(2): 161–169.

Feng ZG and Shi P (2017) Sliding mode control of singular stochastic Markov jump systems. *IEEE Transactions on Automatic Control* 62(8): 4266–4273.

Han YQ (2018) Output-feedback adaptive tracking control of stochastic nonlinear systems using multi-dimensional Taylor network. *International Journal of Adaptive Control and Signal Processing* 32(3): 494–510.

Han YQ (2020a) Adaptive tracking control for a class of stochastic non-linear systems with input saturation constraint using multi-dimensional Taylor network. *IET Control Theory & Applications* 14(9): 1193–1199.

Han YQ (2020b) Adaptive tracking control for a class of stochastic non-linear systems with input delay: A novel approach based on multi-dimensional Taylor network. *IET Control Theory & Applications* 14(15): 2147–2153.

Han YQ and Yan HS (2018) Adaptive multi-dimensional Taylor network tracking control for SISO uncertain stochastic non-linear systems. *IET Control Theory & Applications* 12(8): 1107–1115.

Han YQ and Yan HS (2020) Observer-based multi-dimensional Taylor network decentralised adaptive tracking control of large-scale stochastic nonlinear systems. *International Journal of Control* 93(7): 1605–1618.

Han YQ, Li N, He WJ and Zhu SL (2021a) Adaptive multi-dimensional Taylor network funnel control of a class of nonlinear systems with asymmetric input saturation. *International Journal of Adaptive Control and Signal Processing* 35(5): 713–726.

Han YQ, Zhu SL, Yang SG and Chu L (2021b) Adaptive multi-dimensional Taylor network tracking control for a class of nonlinear systems. *International Journal of Control* 94(2): 277–285.

Han YQ, Zhu SL and Yang SG (2018) Adaptive multi-dimensional Taylor network tracking control for a class of stochastic nonlinear systems with unknown input dead-zone. *IEEE Access* 6: 34543–34554.

Hong YG (2002) Finite-time stabilization and stabilizability of a class of controllable systems. *Systems & Control Letters* 46(4): 231–236.

Hong YR, Huang J and Xu YS (2001) On an output feedback finite-time stabilization problem. *IEEE Transactions on Automatic Control* 46(2): 305–309.

Hua CC, Zhang LL and Guan XP (2015) Decentralized output feedback adaptive NN tracking control for time-delay stochastic nonlinear systems with prescribed performance. *IEEE Transactions on Neural Networks and Learning Systems* 26(11): 2749–2759.

Ji HB and Xi HS (2006) Adaptive output-feedback tracking of stochastic nonlinear systems. *IEEE Transactions on Automatic Control* 51(2): 355–360.

Kamalamiri A, Shahrokhi M and Mohit M (2020) Adaptive finite-time neural control for non-strict feedback systems subject to output constraint, unknown control direction, and input nonlinearities. *Information Sciences* 520: 271–291.

Khoo SY, Yin JL, Man ZH and Yu XH (2013) Finite-time stabilization of stochastic nonlinear systmes in strict-feedback form. *Automatica* 49(5): 1403–1410.

Li J, Chen WS, Li JM and Fang YQ (2009) Adaptive NN output-feedback stabilization for a class of stochastic nonlinear strict-feedback systems. *ISA Transactions* 48(4): 468–475.

Li JM and Yue HY (2015) Adaptive fuzzy tracking control for stochastic nonlinear systems with unknown time-varying delays. *Applied Mathematics and Computation* 256: 514–528.

Li YM, Sui S and Tong SC (2017) Adaptive fuzzy control design for stochastic nonlinear switched systems with arbitrary switchings and unmodeled dynamics. *IEEE Transactions on Cybernetics* 47(2): 403–414.

Liu YC and Zhu QD (2020) Adaptive neural network finite-time tracking control of full state constrained pure feedback stochastic nonlinear systmes. *Journal of the Franklin Institute* 357(11): 6738–6759.

Liu Z, Wang F, Zhang Y and Chen CLP (2016) Fuzzy adaptive quantized control for a class of stochastic nonlinear uncertain systems. *IEEE Transactions on Cybernetics* 46(2): 524–534.

Ma H, Zhou Q, Bai L and Liang HJ (2019) Observer-based adaptive fuzzy fault-tolerant control for stochastic nonstrict-feedback nonlinear systems with input quantization. *IEEE Transactions on Systems, Man, and Cybernetics: Systems* 49(2): 287–298.

Pan ZG and Basar T (1999) Backstepping controller design for nonlinear stochastic systems under a rish-sensitive cost sriterion. *SIAM Journal on Control and Optimization* 37(3): 957–995.

Song ZB and Zhai JY (2017) Finite-time adaptive control for a class of switched stochastic uncertain nonlinear systems. *Journal of the Franklin Institute* 354(12): 4637–4655.

Wang F, Chen B, Liu XP and Lin C (2018) Finite-time adaptive fuzzy tracking control design for nonlinear systems. *IEEE Transactions on Fuzzy Systems* 26(3): 1207–1216.

Wang F, Chen B, Sun YM and Lin C (2019) Finite time control of switched stochastic nonlinear systems. *Fuzzy Sets and Systems* 365: 140–152.

Wang F, Liu Z and Lai GY (2015) Fuzzy adaptive control of nonlinear uncertain plants with unknown dead zone output. *Fuzzy Sets and Systems* 263: 27–48.

Wang HQ, Bai W and Liu PXP (2019) Finite-time adaptive fault-tolerant control for nonlinear systems with multiple faults. *IEEE/CAA Journal of Automatica Sinica* 6(6): 1417–1427.

Wang HQ, Chen B, Liu XP, Liu KF and Lin C (2013) Robust adaptive fuzzy tracking control for pure-feedback stochastic nonlinear systems with input constraints. *IEEE Transactions on Cybernetics* 43(6): 2093–2104.

Wang HQ, Chen B, Liu XP, Liu KF and Lin C (2014) Adaptive neural tracking control for stochastic nonlinear strict-feedback systems with unknown input saturation. *Information Sciences* 269: 300–315.

Wei XJ, Dong LW, Zhang HF, et al. (2019) Adaptive disturbance observer-based control for stochastic systems with multiple heterogeneous disturbances. *International Journal of Robust and Nonlinear Control* 29(16): 5533–5549.

Wu ZJ, Yang J and Shi P (2010) Adaptive tracking for stochastic nonlinear systems with Markovian switching. *IEEE Transactions on Automatic Control* 55(9): 2135–2141.

Yan HS, Han YQ and Sun QM (2018a) Optimal output-feedback tracking of SISO stochastic nonlinear systems using multi-dimensional Taylor network. *Transactions of the Institute of Measurement and Control* 40(10): 3049–3058.

Yan HS, Sun QM and Zhou B (2018b) Multidimensional Taylor network optimal control of SISO nonlinear systems for tracking by output feedback. *Optimal Control Applications and Methods* 39(2): 919–932.

Yang HJ, Wang YJ and Yang YN (2017) Adaptive finite-time control for high-order nonlinear systems with mismathed disturbancees. *International Journal of Adaptive Control and Signal Processing* 31(9): 1296–1307.

Yao LN, Li LF, Guan YC and Wang H (2019) Fault diagnosis and fault-tolerant control for non-Gaussian nonlinear stochastic systems via entropy optimisation. *International Journal of Systems Science* 50(13): 2552–2564.

Zhang J, Xia JW, Sun W, et al. (2018) Finite-time tracking control for stochastic nonlinear systems with full state constraints. *Applied Mathematics and Computation* 338: 207–220.

Zhao XD, Shi P, Zheng XL and Zhang LX (2015) Adaptive tracking control for switched stochastic nonlinear systems with unknown actuator dead-zone. *Automatica* 60: 193–200.

Zhong XN, He HB, Zhang HG and Wang ZS (2015) A neural network based online learning and control approach for Markov jump systems. *Neurocomputing* 149: 116–123.

Zhou B and Yan HS (2014a) A dynamics model based on intermittent feedback multi-dimensional taylor network model. *Acta Automatica Sinica* 40(7): 1517–1521.

Zhou B and Yan HS (2014b) Nonlinear system identification and prediction based on dynamics cluster multi-dimensional Taylor network model. *Control and Decision* 29(1): 33–38.

Zhou Q, Shi P, Liu HH and Xu SY (2012) Neural-network-based decentralized adaptive output-feedback control for large-scale stochastic nonlinear systems. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)* 42(6): 1608–1619.

Zhou Q, Wang LJ, Wu CW and Li HY (2017) Adaptive fuzzy tracking control for a class of pure-feedback nonlinear systems with time-varying delay and unknown dead zone. *Fuzzy Sets and Systems* 329: 36–60.

Zhu SL, Duan DY, Chu L, et al. (2020) Adaptive multi-dimensional Taylor network tracking control for a class of switched nonlinear systems with input nonlinearity. *Transactions of the Institute of Measurement and Control* 42(13): 2482–2491.

Zhu SL, Zhao Y, Li QL, et al. (2021) Short-term traffic flow prediction with wavelet and multi-dimensional Taylor network model. *IEEE Transactions on Intelligent Transportation Systems* 22(5): 3203–3208.