Adv. Appl. Clifford Algebras (2018) 28:87

0188-7009/050001-10
published online September 18, 2018
https://doi.org/10.1007/s00006-018-0905-4

Advances in Applied Clifford Algebras

Linear Map Preserving the Right Spectrum of Quaternion Matrices

Deyu Duan, Xiang Gong, Geng Yuan and Fahui Zhai*

Communicated by Hongbo Li

Abstract. Let $n \geq 2$ and Φ be a right quaternion linear map from the set of $n \times n$ matrices over the quaternion division ring into itself such that the set of $n \times n$ matrices over the complex field is invariant for Φ. We show that Φ preserves the right spectrum of quaternion matrices if and only if there exists an invertible $n \times n$ complex matrix X such that $\Phi(A) = XAX^{-1}$ for every $n \times n$ quaternion matrix A.

Mathematics Subject Classification. Primary 15A04; Secondary 15A18.

Keywords. Linear preserving map, Quaternion matrix, Right spectrum.

1. Introduction

Linear preserver problems are questions about characterizing linear maps on rings or algebras that preserve certain properties, which also are a very old and active research area in matrix and operator theory. For example, in 1897, Frobenius [7] described that the linear map preserves the determinant of complex matrices; in 1949, Dieudonné [5] characterized the form of linear map from the set of singular complex matrices into itself; in 1959, Marcus and Moyls [15] characterized linear maps preserving invertibility of complex matrices. Since 1960, much of the work on the spectrum preserving linear maps have been done by many mathematicians, such as, Aupetit [2], Alaminos, Brešr, Šmrlb and Villena [3], Costara [4], Jafarian and Sourour [11], Li and Tsing [13] and Marcus [14], etc. For some surveys about linear preserver problems and spectrum preserving linear maps, the reader can refer to [10,16] and [19].

This work was supported in part by the project of key research and development planning of Shandong province in China. (Grant No. 2015GGX101020).

*Corresponding author.

The quaternions were discovered by Sir William Rowan Hamilton [8] in 1835. Up to now, quaternions and quaternion matrices have become increasingly useful for practitioners in theory and applications. For example, a number of research papers related to quaternion or quaternion matrix appear in mathematical or physical journals, and quantum mechanics based on quaternion analysis is the main stream in physics. For a detailed account of quaternion, quaternion matrix and their applications, the reader can consult [1] and [17].

Because of noncommutativity of quaternions, many properties of quaternion matrices are different from that of complex matrices, for example, Zhang [21, Example 5.2] illustrated that there exists a 2×2 quaternion matrix admitting an infinite right spectrum, Rodman [17, p. 112] pointed out that two complex matrices which are similar over the quaternion division ring are generally not similar over complex field. Thus a linear map preserving eigenvalues of complex matrices may not be valid for the right spectrum of quaternion matrices. So we hope to characterize the form of linear map preserving the right spectrum of quaternion matrices, which will be discussed in this paper.

Considering that the form of the linear map preserving eigenvalues of complex matrices, and two complex matrices which are similar over the quaternion division ring are generally not similar over complex field, thus we suppose that the set of $n \times n$ complex matrices is invariant for the linear map preserving the right spectrum of quaternion matrices. With this assumption, we characterize the form of a linear map preserving the right spectrum of quaternion matrices. The obtained result is analogous to one of the forms of the linear map preserving eigenvalues of complex matrices.

2. Preliminaries

Throughout this paper, let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex number field, respectively. The quaternion division ring over $\mathbb{R}$, denoted by $\mathbb{H}$, is the set of all elements with the form $a_0 + a_1 i + a_2 j + a_3 k$, where a_0, a_1, a_2 and $a_4 \in \mathbb{R}$, moreover,

$$i^2 = j^2 = k^2 = ijk = -1;$$
$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

If $a = a_0 + a_1 i + a_2 j + a_3 k$, let $\overline{a} = a_0 - a_1 i - a_2 j - a_3 k$ be the conjugate of a. It is clear that $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$, and the multiplication operation of quaternions is noncommutative in $\mathbb{H}$.

Let $M_n(\mathbb{C})$ and $M_n(\mathbb{H})$ denote the set of $n \times n$ matrices over $\mathbb{C}$ and $\mathbb{H}$, respectively. Clearly, $M_n(\mathbb{C}) \subset M_n(\mathbb{H})$. Let $E \in M_n(\mathbb{C})$ denote the identity matrix, $E_{ij} \in M_n(\mathbb{C})$ the matrix whose $(i, j)th$ entry is 1 and other entries are zero, and $E(i,\, j) \in M_n(\mathbb{C})$ be obtained from the identity matrix E by interchanging the ith row (column) and the jth row (column). For $A = [a_{ij}] \in M_n(\mathbb{H})$, let $\overline{A} = [\overline{a_{ij}}]$ be the conjugate of A and $A^t = [a_{ji}]$ the transpose of A. If $A \in M_n(\mathbb{C})$, we write $\mathrm{tr}_{\mathbb{C}}(A)$, $\det_{\mathbb{C}}(A)$ and $\sigma_p(A)$ as the trace, the determinant and the set of distinct complex eigenvalues of complex matrix

A, respectively. Note that every quaternion matrix A can be uniquely written as $A = A_1 + A_2 j$ with A_1 and $A_2 \in M_n(\mathbb{C})$. We shall call the following $2n \times 2n$ complex matrix

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$$

uniquely determined by A, the *complex adjoint* of A, denoted by χ_A.

Due to the noncommutativity of quaternions, there are right and left eigenvalues for a quaternion matrix, right and left quaternion linear maps. This paper only concerns about the right eigenvalue and right quaternion linear map. Their definitions are given below, and the introductions to the left eigenvalue and left quaternion linear map are omitted. The reader can refer to [6,17,21] for more information about eigenvalues and linear maps related to the quaternion matrix.

Definition 2.1. [17,21] Let $A \in M_n(\mathbb{H})$, a $\lambda \in \mathbb{H}$ is called a right eigenvalue of A if $Ax = x\lambda$ for some nonzero $x \in \mathbb{H}^n$, where $\mathbb{H}^n$ denotes the set of vectors of n components over $\mathbb{H}$. The set of distinct right eigenvalues is called to be the right spectrum of A, denoted $\sigma_r(A)$.

Definition 2.2. [6,17] A map $\Phi : M_n(\mathbb{H}) \to M_n(\mathbb{H})$ is said to be the right quaternion linear map if Φ satisfies $\Phi(A + B) = \Phi(A) + \Phi(B)$ and $\Phi(Aq) = \Phi(A)q$ for all $A, B \in M_n(\mathbb{H})$ and $q \in \mathbb{H}$.

When $\Phi : M_n(\mathbb{H}) \to M_n(\mathbb{H})$ is a right quaternion linear map, we say that $M_n(\mathbb{C})$ is *invariant* for Φ if $\Phi(M_n(\mathbb{C})) \subseteq M_n(\mathbb{C})$, and we also say that Φ *preservers right spectrum* of quaternion matrices if $\sigma_r(\Phi(A)) = \sigma_r(A)$ for every $A \in M_n(\mathbb{H})$.

3. The Linear Map Preserving the Right Spectrum

The following Theorem 3.10 characterizes the form of a right quaternion linear map Φ preserving right spectrum of quaternion matrices when the set of complex matrices is invariant for Φ. Before starting the proof of Theorem 3.10, we firstly give the following lemmas.

Lemma 3.1. ([18, Proposition II.1.1.2, 1.1.4]) *Let $A \in M_n(\mathbb{H})$, then*

$$\sigma_r(A) = \mathrm{Sim}(\sigma_p(\chi_A)) \text{ and } \sigma_p(\chi_A) = \sigma_r(A) \cap \mathbb{C},$$

where $\mathrm{Sim}(\sigma_p(\chi_A)) = \{q^{-1}aq : q \in \mathbb{H}, q \neq 0, a \in \sigma_p(\chi_A)\}$.

Lemma 3.2. ([17, Propsition 5.5.2 (2)], [18, Proposition II.1.1(iii)]) *Let $A, X \in M_n(\mathbb{H})$ and X be an invertible matrix, then $\sigma_r(XAX^{-1}) = \sigma_r(A)$.*

Lemma 3.3. ([15, Theorem 3 (ii), (iii)]) *Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map, then $\sigma_p(A) = \sigma_p(\Phi(A))$ for all $A \in M_n(\mathbb{C})$ if and only if there exists an invertible matrix $X \in M_n(\mathbb{C})$ such that*

$$\Phi(A) = XAX^{-1} \text{ or } \Phi(A) = XA^tX^{-1}.$$

Lemma 3.4. *Let Φ be a right quaternion linear map from $M_n(\mathbb{H})$ into itself such that $\sigma_r(A) = \sigma_r(\Phi(A))$ for every $A \in M_n(\mathbb{H})$. If $M_n(\mathbb{C})$ is invariant for Φ, then $\sigma_p(A) = \sigma_p(\Phi(A))$ for every $A \in M_n(\mathbb{C})$.*

Proof. Let $A \in M_n(\mathbb{C})$. Since $M_n(\mathbb{C})$ is invariant for Φ, we have $\Phi(A) \in M_n(\mathbb{C})$. Note that $\sigma_r(A)=\sigma_r(\Phi(A))$, by Lemma 3.1, then $\sigma_p(\chi_A)=\sigma_p(\chi_{\Phi(A)})$. By the matrix representations of χ_A and $\chi_{\Phi(A)}$, then

$$\sigma_p(A) \cup \sigma_p(\overline{A}) = \sigma_p(\Phi(A)) \cup \sigma_p(\overline{\Phi(A)}).$$

Using the properties of the characteristic polynomial of matrix, we imply that $\sigma_p(A) = \sigma_p(\Phi(A))$. □

Remark 3.5. The assumption that $M_n(\mathbb{C})$ is invariant for Φ is indispensable to the conclusion of Lemma 3.4. For example, let

$$X = diag(j,\, 1), \;\text{ and }\; \Phi(A) = XAX^{-1}$$

for every $A \in M_2(\mathbb{H})$. Clearly, $\Phi(E_{21}) \notin M_2(\mathbb{C})$.

By Lemma 3.2, then $\sigma_r(A) = \sigma_r(\Phi(A))$ for every $A \in M_2(\mathbb{H})$. Take $A = diag(1+i,\; 0)$, then $\Phi(A) = diag(1-i,\; 0)$. Since A, $\Phi(A) \in M_2(\mathbb{C})$,

$$\sigma_p(A) = \{1+i,\; 0\},\;\; \sigma_p(\Phi(A)) = \{1-i,\; 0\},$$

we have $\sigma_p(A) \neq \sigma_p(\Phi(A))$. Consequently, $\sigma_p(A) = \sigma_p(\Phi(A))$ is not valid for all $A \in M_2(\mathbb{C})$.

Lemma 3.6. *Let $A, B \in M_n(\mathbb{C})$. If*

$$\sigma_p\left(\begin{bmatrix} 0 & A \\ -\overline{A} & 0 \end{bmatrix}\right) = \sigma_p\left(\begin{bmatrix} 0 & B \\ -\overline{B} & 0 \end{bmatrix}\right), \tag{3.1}$$

then $\sigma_p(\overline{A}\,A) = \sigma_p(\overline{B}B)$.

Proof. By [9, Corollary 4.6.15], there exist $n\times n$ nonsingular complex matrices S and X, and $n \times n$ real matrices R and T such that

$$A = SR\overline{S}^{\,-1},\;\; B = XT\overline{X}^{\,-1}. \tag{3.2}$$

Thus

$$\begin{aligned} \overline{A}A &= \overline{SR\overline{S}^{\,-1}}SR\overline{S}^{\,-1} = \overline{S}R^2\overline{S}^{\,-1},\\ \overline{B}B &= \overline{XT\overline{X}^{\,-1}}XT\overline{X}^{\,-1} = \overline{X}T^2\overline{X}^{\,-1}. \end{aligned} \tag{3.3}$$

By the equality (3.2), then

$$\begin{aligned} \begin{bmatrix} 0 & A \\ -\overline{A} & 0 \end{bmatrix} &= S_1R_1S_1^{-1},\\ \begin{bmatrix} 0 & B \\ -\overline{B} & 0 \end{bmatrix} &= X_1T_1X_1^{-1}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} S_1 &= \begin{bmatrix} S & 0 \\ 0 & \overline{S} \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & R \\ -R & 0 \end{bmatrix},\\ X_1 &= \begin{bmatrix} X & 0 \\ 0 & \overline{X} \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & T \\ -T & 0 \end{bmatrix}. \end{aligned}$$

By the equalities (3.1) and (3.4), then $\sigma_p(R_1) = \sigma_p(T_1)$. Thus $\sigma_p(R_1^2) = \sigma_p(T_1^2)$. Notice that

$$R_1^2 = \begin{bmatrix} -R^2 & 0 \\ 0 & -R^2 \end{bmatrix},\ T_1^2 = \begin{bmatrix} -T^2 & 0 \\ 0 & -T^2 \end{bmatrix},$$

$\sigma_p(R_1^2) = \sigma_p(-R^2)$ and $\sigma_p(T_1^2) = \sigma_p(-T^2)$, then $\sigma_p(R^2) = \sigma_p(T^2)$. By the equality (3.3), we have $\sigma_p(\overline{A}\,A) = \sigma_p(\overline{B}B)$. □

Lemma 3.7. *Let $A, T \in M_n(\mathbb{C})$. If*

$$\sigma_p\left(\begin{bmatrix} 0 & A \\ -\overline{A} & 0 \end{bmatrix}\right) = \sigma_p\left(\begin{bmatrix} 0 & AT \\ -\overline{A}\,\overline{T} & 0 \end{bmatrix}\right), \tag{3.5}$$

then $\sigma_p(\overline{A}\,A) = \sigma_p(\overline{A}\,\overline{T}AT)$, furthermore, if the equality (3.5) is valid for all $A \in M_n(\mathbb{C})$, then there exists $\theta \in [0, 2\pi)$ such that $T = e^{i\theta}E$.

Proof. Apply Lemma 3.6 to the equality (3.5), we have

$$\sigma_p(\overline{A}A) = \sigma_p(\overline{A}\,\overline{T}AT). \tag{3.6}$$

In addition, if the equality (3.5) is valid for all $A \in M_n(\mathbb{C})$, take $A = E_{ij}$, by the equality (3.6), then

$$\sigma_p(\overline{E_{ij}}E_{ij}) = \sigma_p(\overline{E_{ij}}\,\overline{T}E_{ij}T),$$

where $i,\ j = 1,\ 2, \ldots, n$. If $i \neq j$, since $\sigma_p(\overline{E_{ij}}\,E_{ij}) = \{0\}$, one has

$$\mathrm{tr}_{\mathbb{C}}(\overline{E_{ij}}\,\overline{T}E_{ij}T) = 0.$$

By simple matrix computation, then $\overline{t_{ji}}t_{ji} = 0$, $t_{ji} = 0$. Hence T is a diagonal matrix, write $T = diag(t_{11}, t_{22}, \ldots, t_{nn})$. If $i = j$, note that $\overline{E_{ii}}\,E_{ii} = E_{ii}$, by the equality (3.6), then

$$\sigma_p(E_{ii}) = \sigma_p(\overline{E_{ii}}\,\overline{T}E_{ii}T),$$

$i = 1, 2, \ldots, n$. Hence, $\{0,\ 1\} = \{0,\ \overline{t_{ii}}t_{ii}\}$, that is

$$\overline{t_{ii}}t_{ii} = 1,\ i = 1, 2, \ldots, n. \tag{3.7}$$

Take $A = E(i, j)$, where $i \neq j$ and $i,\ j = 1, 2, \ldots, n$. By the equality (3.7), then

$$\sigma_p(\overline{E(i,j)}\,\overline{T}E(i,j)T) = \{\overline{t_{jj}}t_{ii}, \overline{t_{ii}}t_{jj}, 1\}.$$

Note that $E(i, j)E(i, j) = E$, by the equality (3.6), we have

$$\sigma_p(E) = \sigma_p(\overline{E(i,j)}\,\overline{T}E(i,j)T). \tag{3.8}$$

Thus, by the equalities (3.8) and (3.7), we have

$$\overline{t_{jj}}t_{ii} = 1 = \overline{t_{ii}}t_{ii}. \tag{3.9}$$

In terms of the equality (3.9), then $t_{ii} = t_{jj}$. Consequently,

$$t_{11} = t_{22} = \cdots = t_{nn}.$$

By equality (3.7), then there exists $\theta \in [0, 2\pi)$ such that $T = e^{i\theta}E$. □

Lemma 3.8. *Let* $A, T \in M_n(\mathbb{C})$. *If*

$$\sigma_p\left(\begin{bmatrix} 0 & A \\ -\overline{A} & 0 \end{bmatrix}\right) = \sigma_p\left(\begin{bmatrix} 0 & A^tT \\ -\overline{A^t}\,\overline{T} & 0 \end{bmatrix}\right), \tag{3.10}$$

then $\sigma_p(\overline{A}A) = \sigma_p(\overline{A^t}\,\overline{T}A^tT)$. *If the equality* (3.10) *is valid for all* $A \in M_n(\mathbb{C})$, *then there exists* $\theta \in [0, 2\pi)$ *such that* $T = e^{i\theta}E$.

Proof. This is analogous to the proof of Lemma 3.7. □

Lemma 3.9. *When* $n \geq 2$, *then there exists* $A \in M_n(\mathbb{H})$ *such that*

$$\sigma_r(A) \neq \sigma_r(A^t).$$

Proof. When $n = 2$, let $a_1 = 1$, $a_2 = -i$, $b_1 = -i$, $b_2 = -1$,

$$B_1 = \begin{bmatrix} a_2 & 0 \\ 0 & a_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}.$$

Write $B = B_1 + B_2 j$, then

$$\chi_B = \begin{bmatrix} B_1 & B_2 \\ -\overline{B_2} & \overline{B_1} \end{bmatrix}, \quad \chi_{B^t} = \begin{bmatrix} B_1^t & B_2^t \\ -\overline{B_2}^t & \overline{B_1}^t \end{bmatrix}.$$

By simple computation, then

$$\det\nolimits_{\mathbb{C}}(\chi_B) = (a_1\overline{a_2} + \overline{b_1}b_2)(a_2\overline{a_1} + \overline{b_2}b_1) = 0,$$
$$\det\nolimits_{\mathbb{C}}(\chi_{B^t}) = (a_1\overline{a_2} + b_1\overline{b_2})(a_2\overline{a_1} + b_2\overline{b_1}) = 4.$$

Thus $0 \in \sigma_p(\chi_B)$ and $0 \notin \sigma_p(\chi_{B^t})$. By Lemma 3.1, then $\sigma_r(B) \neq \sigma_r(B^t)$. Take $A = B$, we have

$$\sigma_r(A) \neq \sigma_r(A^t).$$

When $n > 2$, take $A = B \oplus I_{n-2}$, then

$$A = (B_1 \oplus I_{n-2}) + (B_2 \oplus 0_{n-2})j.$$

where I_{n-2} and 0_{n-2} denote the identity matrix and the zero-matrix of order $n-2$, respectively. Let $B_1 \oplus I_{n-2} = C_1$ and $B_2 \oplus 0_{n-2} = C_2$, then

$$\chi_A = \begin{bmatrix} C_1 & C_2 \\ -\overline{C_2} & \overline{C_1} \end{bmatrix}, \quad \chi_{A^t} = \begin{bmatrix} C_1^t & C_2^t \\ -\overline{C_2}^t & \overline{C_1}^t \end{bmatrix}.$$

Using [12, p. 33, Theorem 1], the cofactor expansion of determinant, we have

$$\det\nolimits_{\mathbb{C}}(\chi_A) = \det\nolimits_{\mathbb{C}}(\chi_B), \ \ \det\nolimits_{\mathbb{C}}(\chi_{A^t}) = \det\nolimits_{\mathbb{C}}(\chi_{B^t}).$$

Note that $0 \in \sigma_p(\chi_B)$, $0 \notin \sigma_p(\chi_{B^t})$ and the above equality, by Lemma 3.1, then

$$\sigma_r(A) \neq \sigma_r(A^t).$$

□

With the above preparations, we can prove following theorem which illustrates that the form of linear map preserving the right spectrum of quaternion matrices is partly analogous to the form of linear map preserving eigenvalues of complex matrices.

Theorem 3.10. *Let* Φ *be a right quaternion linear map from* $M_n(\mathbb{H})$ *into itself such that* $M_n(\mathbb{C})$ *is invariant for* Φ*. Then* $\sigma_r(A) = \sigma_r(\Phi(A))$ *for every* $A \in M_n(\mathbb{H})$ *if and only if there exists an invertible matrix* $X \in M_n(\mathbb{C})$ *such that* $\Phi(A) = XAX^{-1}$ *for every* $A \in M_n(\mathbb{H})$*.*

Proof. The sufficiency follows immediately from Lemma 3.2. In the following, we give the proof of necessity.

Note that $M_n(\mathbb{C})$ is invariant for Φ, by Lemma 3.4, then

$$\sigma_p(A) = \sigma_p(\Phi(A))$$

for every $A \in M_n(\mathbb{C})$. Apply Lemma 3.3 to the above equality, then there exists an invertible matrix $B \in M_n(\mathbb{C})$ such that

$$\Phi(A) = BAB^{-1} \ \text{ or } \ \Phi(A) = BA^tB^{-1}$$

for all $A \in M_n(\mathbb{C})$.

Since $A \in M_n(\mathbb{H})$ can be uniquely expressed as $A = A_1 + A_2j$, where $A_1,\, A_2 \in M_n(\mathbb{C})$, observe that Φ is a right quaternion linear map, we have

$$\Phi(A) = \Phi(A_1) + \Phi(A_2)j. \tag{3.11}$$

Case 1 If $\Phi(A) = BAB^{-1}$ for all $A \in M_n(\mathbb{C})$.

By the equality (3.11), then

$$\Phi(A) = BA_1B^{-1} + BA_2B^{-1}j = B(A_1 + A_2B^{-1}jB)B^{-1}$$

for $A = A_1 + A_2j$, where $A_1,\, A_2 \in M_n(\mathbb{C})$.

Since $B \in M_n(\mathbb{C})$, we have $jB = \overline{B}j$ and

$$\Phi(A) = B(A_1 + A_2B^{-1}\overline{B}j)B^{-1} \tag{3.12}$$

for $A = A_1 + A_2j$, where $A_1,\, A_2 \in M_n(\mathbb{C})$.

Let $T = B^{-1}\overline{B}$, then $T \in M_n(\mathbb{C})$. Take $A_1 = 0$ in the equality (3.12), by the assumption $\sigma_r(A) = \sigma_r(\Phi(A))$ for all $A \in M_n(\mathbb{H})$, and Lemma 3.2, we obtain that

$$\sigma_r(A_2j) = \sigma_r(A_2Tj)$$

for all $A_2 \in M_n(\mathbb{C})$. By Lemma 3.1, then $\sigma_p(\chi_{A_2j}) = \sigma_p(\chi_{A_2Tj})$ for all $A_2 \in M_n(\mathbb{C})$, that is

$$\sigma_p\left(\begin{bmatrix} 0 & A_2 \\ -\overline{A_2} & 0 \end{bmatrix}\right) = \sigma_p\left(\begin{bmatrix} 0 & A_2T \\ -\overline{A_2}\,\overline{T} & 0 \end{bmatrix}\right)$$

for all $A_2 \in M_n(\mathbb{C})$. Apply Lemma 3.7 to the above equality, then there exists $\theta \in [0, 2\pi)$ such that $T = e^{i\theta}E$. Again use the equality (3.12), we have

$$\Phi(A) = B(A_1 + e^{i\theta}A_2j)B^{-1}.$$

Since $A_1,\, A_2 \in M_n(\mathbb{C})$ and $e^{i\frac{\theta}{2}}j = je^{-i\frac{\theta}{2}}$, one has

$$\begin{aligned} B(A_1 + e^{i\theta}A_2j)B^{-1} &= B(e^{i\frac{\theta}{2}}A_1e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}A_2e^{i\frac{\theta}{2}}j)B^{-1} \\ &= B(e^{i\frac{\theta}{2}}A_1e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}A_2je^{-i\frac{\theta}{2}})B^{-1} \\ &= e^{i\frac{\theta}{2}}B(A_1 + A_2j)e^{-i\frac{\theta}{2}}B^{-1} \\ &= e^{i\frac{\theta}{2}}B(A_1 + A_2j)(e^{i\frac{\theta}{2}}B)^{-1}. \end{aligned}$$

Let $X = e^{i\frac{\theta}{2}}B$, consequently, $\Phi(A) = XAX^{-1}$.

Case 2. If $\Phi(A) = BA^tB^{-1}$ for all $A \in M_n(\mathbb{C})$.

Again note that $B \in M_n(\mathbb{C})$ and $jB = \overline{B}j$, use the equality (3.11), then

$$\Phi(A) = B(A_1^t + A_2^t B^{-1}\overline{B}j)B^{-1}$$

for $A = A_1 + A_2 j$, where $A_1,\ A_2 \in M_n(\mathbb{C})$.

Let $T = B^{-1}\overline{B}$, similar to the arguments of Case 1, by Lemma 3.8, we have $T = e^{i\theta}E$. Hence

$$\Phi(A) = B(A_1^t + e^{i\theta}A_2^t j)B^{-1}.$$

Since

$$\begin{aligned}\chi_{A_1^t + e^{i\theta}A_2^t j} &= \begin{bmatrix} A_1^t & A_2^t e^{i\theta} \\ -\overline{A_2^t}e^{-i\theta} & \overline{A_1^t} \end{bmatrix} \\ &= \begin{bmatrix} E & 0 \\ 0 & e^{-i\theta}E \end{bmatrix} \begin{bmatrix} A_1^t & A_2^t \\ -\overline{A_2^t} & \overline{A_1^t} \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & e^{i\theta}E \end{bmatrix}.\end{aligned}$$

we have $\sigma_p(\chi_{A_1^t + e^{i\theta}A_2^t j}) = \sigma_p(\chi_{A_1^t + A_2^t j})$. Consequently,

$$\mathrm{Sim}(\sigma_p(\chi_{A_1^t + e^{i\theta}A_2^t j})) = \mathrm{Sim}(\sigma_p(\chi_{A_1^t + A_2^t j})).$$

By Lemma 3.1, then $\sigma_r(\Phi(A)) = \sigma_r(A^t)$. Note that the assumption $\sigma_r(A) = \sigma_r(\Phi(A))$ for all $A \in M_n(\mathbb{H})$, we have

$$\sigma_r(A) = \sigma_r(A^t) \tag{3.13}$$

for all $A \in M_n(\mathbb{H})$. By Lemma 3.9, a contradiction is yielded. Consequently, the case 2 is impossible to arise.

By the above arguments, the proof is completed. □

Acknowledgements

The authors are indebted to the Referee's for giving many valuable comments and pointing to us Lemma 3.6 and its proof. Their suggestions have improved the final version of this article.

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Deyu Duan and Xiang Gong
Research Center of Mathematical Modeling
Qingdao University of Science and Technology
Qingdao 266061
China
e-mail: 1215769486@qq.com

Xiang Gong
e-mail: lping1967@163.com

Geng Yuan and Fahui Zhai
Department of Mathematics
Qingdao University of Science and Technology
Qingdao 266061
China
e-mail: fahuiz@163.com

Geng Yuan
e-mail: 2106249719@qq.com

Received: March 12, 2018.
Accepted: September 5, 2018.