

A HYBRID FRACTIONAL-DERIVATIVE AND PERIDYNAMIC MODEL FOR WATER TRANSPORT IN UNSATURATED POROUS MEDIA

YUANYUAN WANG,^{*,†,§} HONGGUANG SUN,^{*,†,||,††} TAO NI,^{‡,§,**,††}
MIRCO ZACCARIOTTO^{§,¶} and UGO GALVANETTO^{§,¶}

**State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering
Hohai University, Nanjing, Jiangsu 210098, P. R. China*

*†College of Mechanics and Materials, Hohai University
Nanjing, Jiangsu 210098, P. R. China*

*‡State Key Laboratory of Geohazard Prevention and Geoenvironment Protection
Chengdu University of Technology, 610059 Chengdu, P. R. China*

*§Industrial Engineering Department, University of Padova
via Venezia 1, Padova 35131, Italy*

*¶Center of Studies and Activities for Space (CISAS)-G. Colombo
via Venezia 15, Padova 35131, Italy*

||shg@hhu.edu.cn

***nitao_sklgp@cdut.edu.cn*

Received March 29, 2023

Accepted May 25, 2023

Published August 18, 2023

Abstract

Richards' equation is a classical differential equation describing water transport in unsaturated porous media, in which the moisture content and the soil matrix depend on the spatial derivative of hydraulic conductivity and hydraulic potential. This paper proposes a nonlocal model and the peridynamic formulation replace the temporal and spatial derivative terms. Peridynamic

^{††}Corresponding authors.

formulation utilizes a spatial integration to describe the path-dependency, so the fast diffusion process of water transport in unsaturated porous media can be captured, while the Caputo derivative accurately describes the sub-diffusion phenomenon caused by the fractal nature of heterogeneous media. A one-dimensional water transport problem with a constant permeability coefficient is first addressed. Convergence studies on the nonlocal parameters are carried out. The excellent agreement between the numerical and analytical solutions validates the proposed model for its accuracy and parameter stability. Subsequently, the wetting process in two porous building materials is simulated. The comparison of the numerical results with experimental observations further demonstrates the capability of the proposed model in describing water transport phenomena in unsaturated porous media.

Keywords: Caputo Derivative; Peridynamics; Nonlocal Model; Fractal Porous Media; Anomalous Diffusion.

1. INTRODUCTION

Soil water movement is an important component of the natural water cycle and the central link of precipitation, soil water and groundwater in the terrestrial water cycle.^{1–4} Naturally formed soil exhibits high shrinkage or expansion potential due to the variations in matric suction.^{5–7} When the soil becomes dry, the tensile stress exceeds the failure limit of the soil, which triggers the formation and evolution of cracks.^{8,9} These cracks increase the connectivity of soil pore, so that water can accelerate the wetting front along the preferential flow paths, resulting in fast diffusion. Meanwhile, fast diffusion leads to the rapid infiltration of nutrients and pesticides from plants into deeper soil layers, reducing the residence time of pollutants in unsaturated areas and increasing the possibility of groundwater or surface water pollution.^{10–14} In addition, when the soil becomes moist and expands, cracks may close during the infiltration phase, resulting in a highly nonlinear response.¹⁵ Therefore, accurate prediction of the water transport process in unsaturated soil has a great relevance for Environmental Protection.

The Richards' equation is the fundamental equation describing the law of soil water transport.¹⁶ It was established in 1931 by Richards, who used Darcy's law to study capillary conduction in porous media.¹⁷ However, the classical Richards' equation only applies to isotropic soil and incompressible fluid and cannot accurately describe the water transport process of anomalous diffusion in water transport.¹⁸ The fractal models proposed by Xu, Moore, and Gallant¹⁹ are useful to describe soil heterogeneity. Indicators of fractal-type soils include power-law distributions of pore space, particle size,

and mass distribution. Tyler and Wheatcraft²⁰ explained how fractal porous media can lead to anomalous diffusion. Then, the nonlocal models are effective tool for describing the anomalous process caused by fractal porous media, the ones based on a continuous time random walk^{21,22} or fractional derivative²³ consider the temporal nonlocality, while the ones based on the fractional Laplace operator²⁴ or peridynamics²⁵ consider spatial nonlocality. Considering the fractional derivative model (FDM) with time memory, Pachepsky, Timlin, and Rawls¹⁸ proposed the fractional Richards' equation, but this type of fractional equation is unsuitable for simulating fast diffusion cases because it is limited to showing sub-diffusion characteristics.^{26,27} Spatial nonlocality is caused by the spatial dependence between points in anomalous diffusions, such as the flux at point x , which depends not only on the local state quantity of x but also on the state quantity of points in the near-range horizon of x . In addition to providing a method to simulate anomalous diffusion processes through nonlocality, the Peridynamic model (PDM) forgoes the dependence on the spatial derivatives to overcome the discontinuity problem of cracks in the field variables. Recently, the PDM has been used to simulate fluid flow in porous media. Katiyar proposed a method to construct a peridynamic formulation for the convection–diffusion problems in heterogeneous porous media.²⁸ Jabakhanji²⁹ established a bond-based PDM for diffusion problems in unsaturated porous media. As an alternative to the classical Richards' equation, the PDM results are consistent with the classical model. It is worth noting that peridynamics is mostly used to study fluid flow

and soil deformation,^{30,31} rather than spatial non-locality.

In this paper, a hybrid fractional-derivative and peridynamic model is proposed based on Richards' equation for the water transport problems in unsaturated porous media. The proposed nonlocal model extends Richards' equation using a time-fractional operator to describe anomalous diffusion.³²⁻³⁴ The spatial term of the Richards' equation is replaced by the peridynamic operator,^{35,36} which can not only solve discontinuous problems but also display non-locality in space.³⁷⁻⁴⁰ The new nonlocal Richards' equation captures fast diffusion and sub-diffusion processes, which makes it possible that this model can describe the fluid-driven evolving nonlocality in highly heterogeneous porous media with the existence of cracks.

The contents of the paper are organized as follows. Section 2 describes the derivations of the hybrid fractional-derivative and peridynamic model. In Sec. 3, the discretization and numerical implementation of the proposed model are introduced in detail. Section 4 presents two numerical examples, and the comparisons among the numerical solutions, analytical solutions, and experimental results demonstrate the accuracy and capabilities of the proposed model. Finally, Sec. 5 concludes the paper.

2. NONLOCAL MODEL FOR UNSATURATED POROUS MEDIA

The classical Richards' equation is based on Darcy's law and is applicable for solving water transport in unsaturated porous media. For one-dimensional problems, the form of Richards' equation can be given as

$$\frac{\partial \theta(x, t)}{\partial t} = \frac{\partial}{\partial x} \left[D(\theta) \frac{\partial \theta(x, t)}{\partial x} \right], \quad (1)$$

where $\theta(x, t)$ is the moisture content, and $D(\theta)$ is the hydraulic conductivity.

Due to the complexity of the structure, the Richards' equation cannot accurately describe water transport process in various porous media. For porous structures with highly connected channels, cracks, or macropores, water transport exhibits deviation from Boltzmann scaling. Thus, the connectivity accelerates the movement of water content along the predominant pathways resulting in fast diffusion. In contrast, low permeability

regions hinder water flow, leading to sub-diffusion. Based on nonlocal models, Pachepsky *et al.*¹⁸ proposed the fractional Richards' equation to describe the power-law distribution for waiting periods, which is given as

$$D_a^\alpha \theta(x, t) = \frac{\partial^\alpha \theta(x, t)}{\partial t^\alpha} = \frac{\partial}{\partial x} \left[D_\alpha(\theta) \frac{\partial \theta(x, t)}{\partial x} \right], \quad (2)$$

where α represents the order for the FDM in the range $0 < \alpha \leq 1$ and $D_\alpha(\theta)$ is the fractional hydraulic conductivity. $D_a^\alpha \theta(x, t)$ represents the Caputo derivative as

$$D_a^\alpha \theta(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_a^t (t - \tau)^{-\alpha} \times \frac{d}{dt} \theta(x, \tau) d\tau, \quad 0 < \alpha \leq 1. \quad (3)$$

Note that if $\alpha = 1$, Eq. (2) will be reduced to the classical Richards' equation.

On the other hand, the time-fractional derivative model cannot accurately describe the spatial nonlocality properties of the porous media with predominant pathways. To overcome this, we introduce a nonlocal integral term based on peridynamic theory to replace the local continuum mechanics term in the classical Richards' equation. Therefore it is the hydro-potential field that drives the flow, which does not depend on any spatial derivative and it is possible to capture the anomalous diffusion process in unsaturated porous media.

A body Ω modeled by PD is assumed to be composed by a series of material points. Supposing that x is a point in Ω , it will interact with all the other points over a prescribed domain \mathcal{H}_x ,²⁵ which is typically a sphere space in 3D or a circle surface in 2D. The neighborhood can be described by a radius of length δ (the horizon) and mathematically defined as

$$\mathcal{H}_x = \{x' \in \Omega | 0 < |x - x'| \leq \delta\}, \quad (4)$$

where x' represents the points located within the neighborhood of point x . $|x' - x|$ is the relative position vector, which can be also used to describe the bond connecting points x and x' . Here, the relative position of point x and x' is $|x - x'| = |x' - x|$, so in order to facilitate a unified notation, we adopt $|x - x'|$ to provide the mathematical definition of the domain \mathcal{H}_x .

When using PD to model the water transport in unsaturated porous media, the water guided by the height difference of the unit head in unit time can

be expressed by the hydraulic conductivity density $C(x, x')$ defined as

$$C(x, x') = \frac{k(x, x')}{|x - x'|}, \quad (5)$$

and $k(x, x')$ is the micro-permeability coefficient in the PD framework derived as

$$k(x, x') = D(\theta)\omega(x, x'), \quad (6)$$

where $D(\theta)$ represents hydraulic conductivity, and $\omega(x, x')$ is influence function. Given the different heights of water heads at points x and x' , marked as $H(x)$ and $H(x')$, then the increment of moisture content at the two points in the unit time via the bond xx' can be calculated by²⁹

$$\Delta\theta_m(x, x') = k(x, x') \frac{[H(x') - H(x)]}{|x - x'|} dV_{x'} dV_x, \quad (7)$$

$$\Delta\theta_m(x', x) = k(x', x) \frac{[H(x) - H(x')]}{|x' - x|} dV_x dV_{x'}, \quad (8)$$

in which dV_x and $dV_{x'}$ represent the infinitesimal volumes occupied by x and x' , respectively. Due to mass conservation, we have $\Delta\theta_m(x, x') = -\Delta\theta_m(x', x)$, and we can get the following restriction $k(\cdot)$:

$$k(x, x') = k(x', x). \quad (9)$$

Finally, considering the external sources $S(x)$, the nonlocal Richards' equation for the water flow in porous media at point x can be given as

$$\frac{\partial^\alpha \theta(x, t)}{\partial t^\alpha} = \int_{\mathcal{H}_x} k(x, x') \frac{[H(x') - H(x)]}{|x - x'|} dV_{x'} + S(x, t). \quad (10)$$

For the PDM, there are two parameters influencing the degree of nonlocality: the horizon δ^{41} and the influence function. Several influence functions have been used in the FDM and PDM to describe the degree of nonlocality.⁴² The kernel function of the FDM is a typical power-law function,⁴³ while the PDM usually takes a Gaussian function as the influence function. Compared to the Gaussian function, the power-law function is more suitable for describing the nonlocality of the model;⁴¹ thus, the kernel function of both models is defined as a power-law kernel function:

$$\omega(x, x') = |x - x'|^{-\beta}. \quad (11)$$

Naturally formed soils are usually heterogeneous, and its hydraulic conductivity differs in space.

Additionally, due to the dependence of hydraulic conductivity on moisture content, even the soils with uniform parameters will exhibit spatial variability when in the unsaturated state. Therefore, to describe such a spatial variability of hydraulic conductivity in heterogeneous unsaturated porous media, an averaging operation is adopted to calculate the bond-level permeability coefficient, taking into account the hydraulic conductivity properties at the two nodes connected by the bond, which is given as

$$k(x, x') = \frac{K(x) + K(x')}{2}, \quad (12)$$

where $K(x)$ and $K(x')$ are the micro-permeability coefficients at points x and x' , respectively, whose values vary with the saturation at the points.

Consequently, in the 1D case, the pairwise interaction of the bond xx' exerted on point x can be obtained as²⁹

$$J(x, x') = \frac{(K(x) + K(x'))}{\delta^2} \frac{(H(x') - H(x))}{|x - x'|}. \quad (13)$$

Then the nonlocal Richards' equation of water transport in unsaturated porous media at point x can be rewritten as

$$\frac{\partial^\alpha \theta(x, t)}{\partial t^\alpha} = \int_{\mathcal{H}_x} |x - x'|^{-\beta} \frac{(K(x) + K(x'))}{\delta^2 |x - x'|} (H(x') - H(x)) dV_{x'} + S(x, t), \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 1. \quad (14)$$

The moisture content θ , matrix potential H , and hydraulic conductivity K of each point are three independent functions that can be obtained by matching the proposed soil moisture curve with the experimental data. Here, the Van Genuchten model⁴⁴ is used to calculate the matrix potential and hydraulic conductivity, and the relevant formulas are given as

$$\theta(H) = \theta_r + \frac{\theta_s - \theta_r}{[1 + (a|H|)^n]^{1-1/n}}, \quad (15)$$

$$K(\theta) = \left(\frac{\theta - \theta_r}{\theta_s - \theta_r} \right)^{\frac{1}{2}} \times \left\{ 1 - \left[1 - \left(\frac{\theta - \theta_r}{\theta_s - \theta_r} \right)^{\frac{n}{n-1}} \right]^{\frac{n-1}{n}} \right\}^2, \quad (16)$$

in which θ_s and θ_r represent the saturation moisture content and residual moisture content, respectively. a is a parameter concerned with the inverse of the air entry pressure, and n is a parameter of the soil pore size. In fact, the matrix potential is a function of the moisture content, rewriting Eq. (15), its expression can be obtained as

$$H(\theta) = -\frac{1}{a} \left[\left(\frac{\theta_s - \theta_r}{\theta - \theta_r} \right)^{\frac{n}{n-1}} - 1 \right]^{\frac{1}{n}} \quad (17)$$

Due to the existence of the terms related to the spatial derivatives of hydraulic conductivity and hydraulic potential, the solution of the classical Richard's equation requires that the hydraulic conductivity field must be differentiable, and the hydraulic potential must be twice differentiable. However, Eqs. (10) and (14) do not rely on any spatial derivatives, which means that the proposed nonlocal Richard's equation is valid anywhere in the computational domain, even in the presence of cracks.

3. NUMERICAL IMPLEMENTATION

It is difficult to obtain analytical solutions concerning nonlocal Richards' equations with different initial and boundary conditions. In this section, the numerical implementation of the proposed nonlocal Richards' equation will be presented in detail.

In the proposed nonlocal Richards' equation, there are three independent functions: $\theta(x_n)$ represents the moisture content, $H(x_n)$ and $K(x_n)$ are the hydraulic potential and hydraulic conductivity at point x_n , respectively. Then, a discretized soil column with grid spacing of Δx and a horizon radius $\delta = m\Delta x$, in which m is the horizon radius. To solve the nonlocal equation, the Caputo derivative is adopted for the fractional derivative term.

Therefore, the discrete forms of the time-fractional term in Eqs. (10) and (14) can be obtained as

$$\begin{aligned} & \frac{\partial^\alpha \theta(x, t_{k+1})}{\partial t^\alpha} \\ & \approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{\theta(x, t_{j+1}) - \theta(x, t_j)}{\tau} \\ & \quad \times \int_{j\tau}^{(j+1)\tau} \frac{d\xi}{(t_{k+1} - \xi)^{-\alpha}} \end{aligned}$$

$$\begin{aligned} & = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^k \frac{\theta(x, t_{k-j+1}) - \theta(x, t_{k-j})}{\tau} \\ & \quad \times [(j+1)^{1-\alpha} - j^{1-\alpha}]. \end{aligned} \quad (18)$$

For $k = 0$, Eq. (18) is reduced to

$$\frac{\partial^\alpha \theta(x_n, t_1)}{\partial t^\alpha} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} [\theta(x_n, t_1) - \theta(x_n, t_0)]. \quad (19)$$

For $k = 1$, Eq. (18) is stated as

$$\begin{aligned} \frac{\partial^\alpha \theta(x_n, t_2)}{\partial t^\alpha} & = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} [\theta(x_n, t_2) \\ & \quad + \theta(x_n, t_1)(2^{1-\alpha} - 2) \\ & \quad - \theta(x_n, t_0)(2^{1-\alpha} - 1)]. \end{aligned} \quad (20)$$

For $k \geq 2$, Eq. (18) is described as

$$\begin{aligned} & \frac{\partial^\alpha \theta(x_n, t_{k+1})}{\partial t^\alpha} \\ & = \frac{\tau^{-\alpha}}{\alpha(2-\alpha)} \theta(x_n, t_{k+1}) + \frac{\tau^{-\alpha}}{\alpha(2-\alpha)} \\ & \quad \times \theta(x_n, t_k)(2^{1-\alpha} - 2) \\ & \quad + \frac{\tau^{-\alpha}}{\alpha(2-\alpha)} \sum_{j=1}^{k-1} \theta(x_n, t_{k-j}) \\ & \quad \times ((j+2)^{1-\alpha} - 2(j+1)^{1-\alpha} + j^{1-\alpha}) \\ & \quad - \frac{\tau^{-\alpha}}{\alpha(2-\alpha)} \theta(x_n, t_0)((k+1)^{1-\alpha} - k^{1-\alpha}). \end{aligned} \quad (21)$$

Additionally, for the right side of Eq. (10), by replacing the power-law function in Eq. (11), a discrete equation for the evolution of moisture content in the 1D cases is obtained as follows:

$$\begin{aligned} & \int_{\mathcal{H}_x} |x - x'|^{-\beta} \frac{3(K(x) + K(x'))}{\delta^2 |x - x'|} \\ & \quad \times (H(x') - H(x)) dV_{x'} \\ & \approx \sum_{\substack{p=n-m \\ p \neq n}}^{n+m} |x_n - x_p|^{-\beta} \frac{3(K(x_n) + K(x_p))}{\delta^2 |x_n - x_p|} \\ & \quad \times (H(x_p) - H(x_n)) \Delta x, \end{aligned} \quad (22)$$

where m is the horizon radius in multiples of grid lengths, and p contains all the points belonging to the neighborhood of point x_n .

For Eq. (14), the first step is

$$\begin{aligned} \theta(x_n, t_1) &= \theta(x_n, t_0) + \frac{\Gamma(2-\alpha)}{\tau^{-\alpha}} \sum_{\substack{p=n-m \\ p \neq n}}^{n+m} |x_n - x_p|^{-\beta} \\ &\times \frac{3(K(x_n, t_0) + K(x_p, t_0))}{\delta^2 |x_n - x_p|} \\ &\times (H(x_p, t_0) - H(x_n, t_0)) \Delta x. \end{aligned} \quad (23)$$

The second step is

$$\begin{aligned} \theta(x_n, t_2) &= -\theta(x_n, t_1)(2^{1-\alpha} - 2) + \theta(x_n, t_0) \\ &\times (2^{1-\alpha} - 1) + \frac{\Gamma(2-\alpha)}{\tau^{-\alpha}} \sum_{\substack{p=n-m \\ p \neq n}}^{n+m} \\ &\times |x_n - x_p|^{-\beta} \frac{3(K(x_n, t_0) + K(x_p, t_0))}{\delta^2 |x_n - x_p|} \\ &\times (H(x_p, t_0) - H(x_n, t_0)) \Delta x. \end{aligned} \quad (24)$$

And the remaining steps are

$$\begin{aligned} \theta(x_n, t_{k+1}) &= -\theta(x_n, t_k)(2^{1-\alpha} - 2) - \sum_{j=1}^{k-1} \theta(x_n, t_{k-j}) \\ &\times ((j+2)^{1-\alpha} - 2(j+1)^{1-\alpha} + j^{1-\alpha}) \\ &+ \theta(x_n, t_0)((k+1)^{1-\alpha} - k^{1-\alpha}) \\ &+ \frac{\Gamma(2-\alpha)}{\tau^{-\alpha}} \sum_{\substack{p=n-m \\ p \neq n}}^{n+m} |x_n - x_p|^{-\beta} \\ &\times \frac{3(K(x_n, t_0) + K(x_p, t_0))}{\delta^2 |x_n - x_p|} \\ &\times (H(x_p, t_0) - H(x_n, t_0)) \Delta x. \end{aligned} \quad (25)$$

A flowchart of the solution algorithm for the diffusion problems in the unsaturated porous media is shown in Fig. 1. Given the initial and boundary conditions, the moisture content value will be calculated interactively according to Eqs. (10) and (14) through explicit time integration. At the end of each time step, the moisture content, matrix potential, and hydraulic conductivity of each point in the computational domain are updated with the new moisture content value.

4. NUMERICAL EXAMPLES

In this section, a 1D anomalous diffusion problem with constant hydraulic conductivity is considered first to validate the proposed model and to select

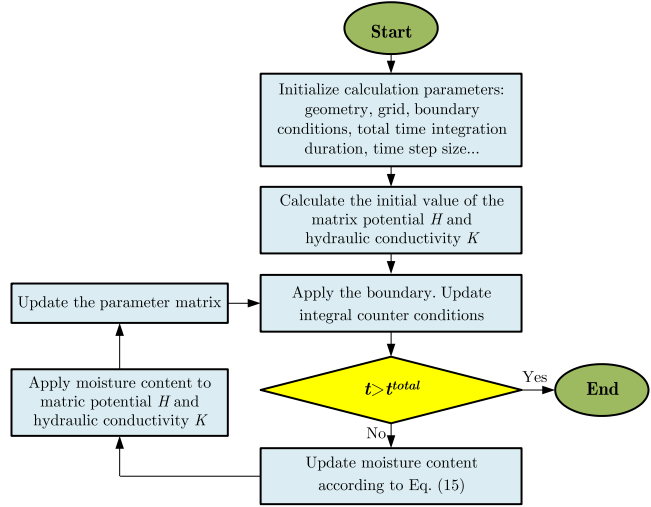


Fig. 1 Flowchart of the solution algorithm for porous media problems.

the most proper nonlocal parameters (the fractional order α and PD horizon radius δ). Subsequently, the proposed hybrid fractional-derivative and peridynamic formulation is used to simulate two sets of 1D anomalous diffusion experiments to further demonstrate its capabilities and accuracy.

4.1. A 1D Diffusion Problem with a Constant Hydraulic Conductivity

The hybrid fractional-derivative and peridynamic governing equation for the anomalous diffusion problem of solute transport in the 1D porous domain with a constant permeability coefficient can be given as

$$\begin{aligned} \frac{\partial^\alpha \theta(x, t)}{\partial t^\alpha} &= \int_{\mathcal{H}_x} k(x, x') \frac{[H(x', t) - H(x, t)]}{|x - x'|} dV_{x'} \\ &+ S(x, t), \quad 0 < \alpha \leq 1. \end{aligned} \quad (26)$$

Here, we use the power-law function as influence function, $k(x, x') = \frac{6}{\delta^2} |x - x'|^{-1}$. And the Dirichlet boundary condition is adopted. The source term is given as

$$S(x, t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 3dx^2. \quad (27)$$

Therefore, the analytical solution for the described problem can be obtained as

$$\theta_{\text{exact}}(x, t) = x^3 + t. \quad (28)$$

The computational domain is $\Omega = [0, 1]$, and a transient source releasing is set at its left boundary, describing a typical pollution process. An explicit

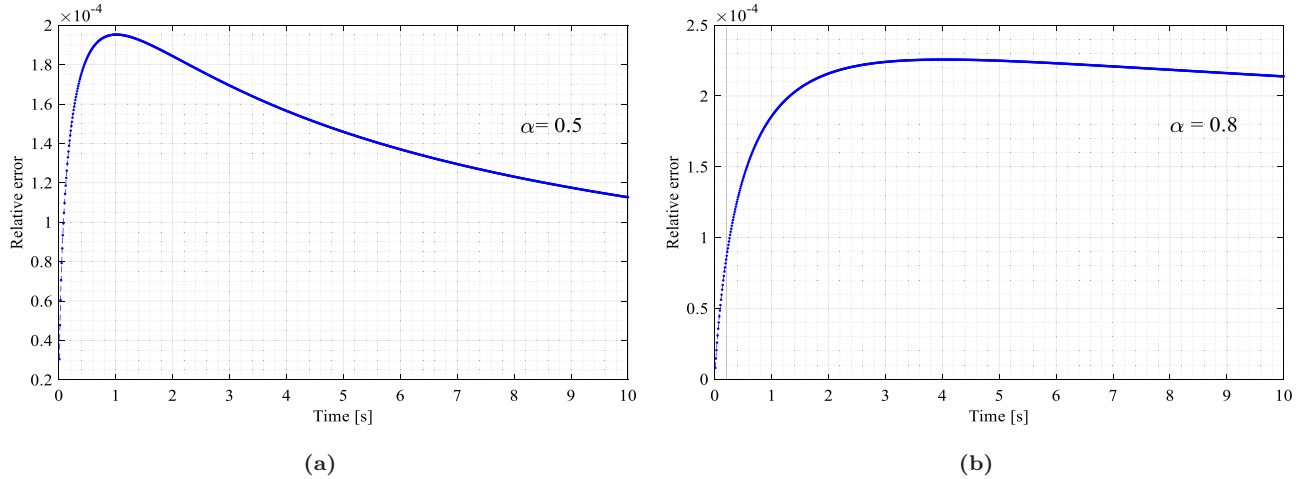


Fig. 2 The curves of relative errors between the nonlocal Richards’ equation and analytical solution with $\alpha = 0.5$ (a) and 0.8 (b), respectively.

time integration scheme is adopted to obtain the numerical solution. The simulation time duration is 10 s, and a fixed time step of $\Delta t = 0.01$ s is taken for the time integration. In the proposed hybrid fractional-derivative and peridynamic model, two parameters describe the nonlocality, the fractional order α and PD horizon radius δ , which could significantly affect the numerical solutions. To select the most suitable model parameters promising the best compromise between accuracy and computational cost,⁴⁵ several numerical cases are designed in the following parts of this section.

For the later use and to measure the accuracy of the numerical solutions, a global relative error is defined as

$$\begin{aligned} \text{Relative Error} &= \frac{\left[\sum_i^N [\theta_{\text{exact}}(x_i, t) - \theta_{\text{num}}(x_i, t)]^2 \right]^{1/2}}{\left[\sum_i^N [\theta_{\text{exact}}(x_i, t)]^2 \right]^{1/2}}. \quad (29) \end{aligned}$$

First, two cases with different fractional orders $\alpha = 0.5$ and $\alpha = 0.8$ are considered to investigate the influence of fractional parameters on the accuracy and numerical stability of the proposed model. In this part, a uniform grid with a spacing of $\Delta x = 1$ mm is used for the spatial discretization, and the m -ratio is taken as $m = 3$, resulting a horizon radius of $\delta = 3$ mm. Figure 2 shows the variations of the global relative error versus time, which describes that the numerical solutions obtained by the hybrid fractional-derivative and peridynamic model agree very well with the analytical solutions. In addition, the relative error values obtained with

different fractional parameters are small in magnitude, indicating that the proposed model is both accurate and stable.

Then, the m -convergence is studied with a fixed fractional order of $\alpha = 0.8$. The discretization parameters used in the convergence studies are taken as $m = 2, 4, 6$ and 8 , $\delta = 1, 3$ and 6 cm, resulting in 12 different cases. The obtained global relative error values at the final time step are shown in Fig. 3. Comparing the results of the cases with the same δ , the relative error value decreases versus the m -ratio increases. In contrast, in the cases with the same m -ratio, the relative error decreases versus the δ value increases. Based on the shown results, $\delta = 3$ mm and $m = 3$ can be economical choices, ensuring an accurate solution of the proposed model applied to the described problem.

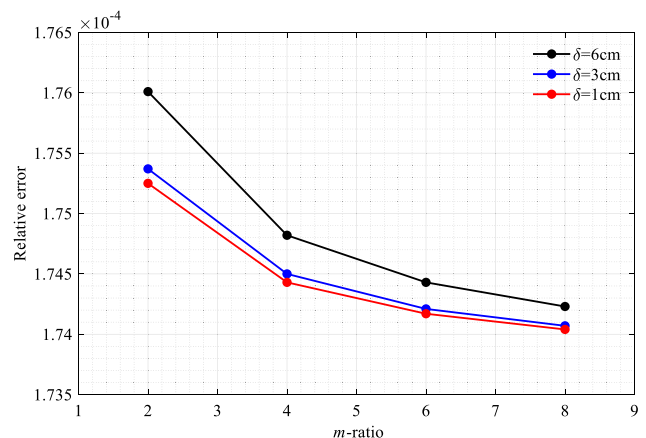


Fig. 3 Convergence studies of the nonlocal model with $\alpha = 0.8$.

4.2. A 1D Anomalous Diffusion Problem in Unsaturated Porous Media

The hybrid fractional-derivative and peridynamic equation for the 1D anomalous diffusion problem in unsaturated media can be given as

$$\frac{\partial^\alpha \theta(x_n, t)}{\partial t^\alpha} = \sum_{\substack{p=n-m \\ p \neq n}}^{n+m} |x_n - x_p|^{-\beta} \frac{3(K(x_n) + K(x_p))}{\delta^2 |x_n - x_p|} \times (H(x_p) - H(x_n)) \Delta x, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 1, \quad (30)$$

where the boundary conditions and initial conditions can be defined as

$$\begin{cases} \theta(x, t) = \theta_s, & x \in \text{left boundary}, t \in [0, T], \\ \theta(x, 0) = \theta_r, & \text{other.} \end{cases} \quad (31)$$

Following El Abd and Milczarek⁴⁶ two sets of anomalous diffusion experiments were carried out with fired-clay and siliceous bricks. The diffusion parameters for the two materials can be determined using the Van Genuchten model, as shown in Table 1. In this subsection, we will use the proposed model to simulate the two experiments to further evaluate its performance and accuracy.

The length of the specimens used in the experiments is 100 mm. In the simulations, the solution domain will be discretized by a uniform grid with the size of $\Delta x = 0.05$ mm. The m -ratio is taken as $m = 3$, and thus the horizon radius will be $\delta = \Delta x \times m = 0.15$ mm. An explicit time integration scheme is adopted with a time-step of $\Delta t = 0.00001$ s. A series of numerical experiments were carried out to determine the most suitable parameters in Eq. (29), they are $\alpha_1 = 1$ and $\beta_1 = 0.8$ for the fired-clay brick, $\alpha_2 = 0.86$ and $\beta_2 = 1$ for the siliceous brick. Figure 4a shows the comparison of the wetting front configurations in the fired-clay brick at $t = 419, 2219, 14879, 26099,$

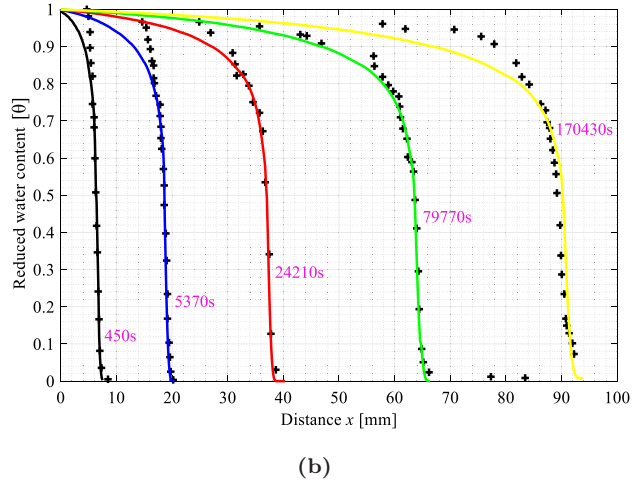
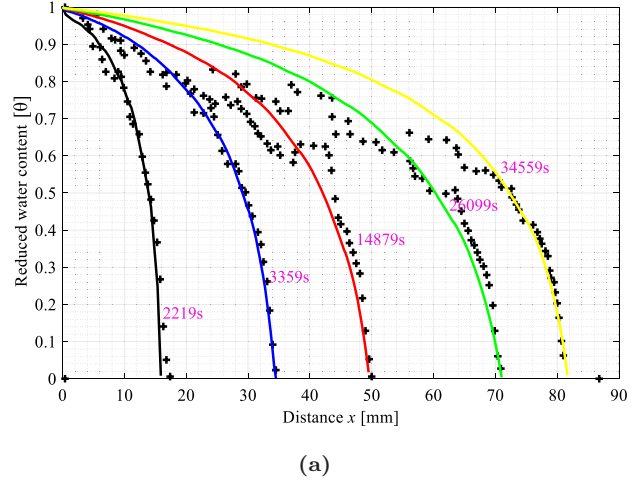


Fig. 4 Comparison of the numerical results and experimental data.⁴⁶ The solid lines are the numerical results for (a) fired-clay brick and (b) siliceous brick.

and 34559 s obtained from the experiment data and the simulation results, while the numerically and experimentally observed wetting front configurations in the siliceous brick at $t = 450, 5370, 24210, 79770,$ and 170430 s are plotted and compared in Fig. 4b. All the numerical solutions agree well with the experimental results, further demonstrating the high accuracy and remarkable ability of the proposed model to describe the anomalous diffusion phenomena in unsaturated porous media.

Table 1 Diffusion Parameters of Fired-clay and Siliceous Materials Determined by the Van Genuchten Model.

Items	θ_s	θ_r	a	n
Fired-clay brick	0.2782	0.1995	0.1637	4.5282
Siliceous brick	0.1907	0.0109	0.1589	15.0214

5. CONCLUSIONS

This paper proposes a generalized nonlocal Richards' equation to characterize anomalous diffusion in unsaturated porous media, which is based on the Caputo derivative and the peridynamic model.

Combined with the spatial correlation of PDM and the time dependence of FDM, the nonlocal Richards' equation can capture both sub-diffusion and fast diffusion phenomena in unsaturated porous media. The numerical results depict that the nonlocal Richards' equation can describe the water transport process in porous media. Two groups of experimental data of wetting process kinetics in fired-clay and siliceous brick were used to test its applicability in characterizing the non-Boltzmann scaling of water transport. All the results show that the proposed nonlocal Richards' equation can accurately solve anomalous diffusion problems. Furthermore, compared to the classical Richards' equation, the proposed nonlocal model exhibits advantages in addressing unsaturated and fractal porous problems.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (Nos. 11972148, and 42207226), the Postgraduate Research and Practice Innovation Program of Jiangsu Province (Grant No. KYCX21_0462). T. Ni, U. Galvanetto and M. Zaccariotto acknowledge the support they received from University of Padua under the research project BIRD2020 NR.202824/20.

REFERENCES

1. V. Girardi *et al.*, Wetting induced instabilities in layered slopes: A material point method analysis, *Eng. Geol.* **313** (2023) 106978.
2. J. Vanderborght *et al.*, Wetting induced instabilities in layered slopes: A material point method analysis, *Water Resour. Res.* **53** (2017) 1057–1079.
3. B. Xiao *et al.*, A fractal model for water flow through unsaturated porous rocks, *Fractals — Compl. Geom. Patterns Scal. Nature Soc.* **26** (2018) 1840015.
4. H. Xu, X. He and D. Sheng, Rainfall-induced landslides from initialization to post-failure flows: Stochastic analysis with machine learning, *Mathematics* **10** (2022) 4426.
5. D. Painuli *et al.*, Crack formation in a swell–shrink soil under various managements, *Agricult. Res.* **6** (2017) 66–72.
6. C. Xie *et al.*, Combined measure of geometry optimization and vegetation for expansive soil slopes, *Comput. Geotech.* **123** (2020) 103588.
7. B. Zamin *et al.*, Development of some novel suction-based correlations for swell behavior of expansive soils, *Adv. Civil Eng.* **2021** (2022) 4825593.
8. Y. Luo *et al.*, Investigation and prediction of water infiltration process in cracked soils based on a full-scale model test, *Geoderma* **400** (2021) 115111.
9. J. Zhang *et al.*, Effects of preferential flow induced by desiccation cracks on slope stability, *Eng. Geol.* **288** (2021) 106164.
10. R. D. Harmel *et al.*, Comparison of nutrient loss pathways: Run-off and seepage flow in Vertisols, *Hydrolog. Process.* **33** (2019) 2384–2393.
11. N. Huang, R. Liu and Y. Jiang, Evaluating the effect of aperture variation on the hydraulic properties of the three-dimensional fractal-like tree network model, *Fract.-Compl. Geom. Patterns Scal. Nature Soc.* **28** (2020) 2050112.
12. Y. Liang *et al.*, Hausdorff derivative model for characterization of non-fickian mixing in fractal porous media, *Fractals — Compl. Geom. Patterns Scal. Nature Soc.* **27** (2019) 1950063.
13. L. Summerton *et al.*, Effects of soil redistribution by tillage on subsequent transport of pesticide to subsurface drains, *Pest Manag. Sci.* **79** (2023) 616–626.
14. W. Xu and Y. Liang, A nonlocal structural derivative model based on the caputo fractional derivative for superfast diffusion in heterogeneous media, *Fractals — Compl. Geom. Patterns Scal. Nature Soc.* **28** (2021) 2050122.
15. J. Wu *et al.*, Fractal characteristics of low-permeability sandstone reservoirs, *Fractals—Compl. Geom. Patterns Scal. Nature Soc.* **30** (2022) 2250075.
16. X. Zeng and M. Decker, Improving the numerical solution of soil moisture–based Richards equation for land models with a deep or shallow water table, *J. Hydrometeor.* **10** (2009) 308–319.
17. L. A. Richards, Capillary conduction of liquids through porous mediums, *Physics* **1** (1931) 318–333.
18. Y. Pachepsky, D. Timlin and W. Rawls, Generalized Richards' equation to simulate water transport in unsaturated soils, *J. Hydrology* **272** (2003) 3–13.
19. T. Xu, I. Moore and J. Gallant, Fractals, fractal dimensions and landscapes—a review, *Geomorphology* **8** (1993) 245–262.
20. S. Tyler and S. Wheatcraft, Fractal processes in soil water retention, *Water Resour. Res.* **26** (1990) 1047–1054.
21. N. Masuda, M. A. Porter and R. Lambiotte, Random walks and diffusion on networks, *Phys. Rep.* **716** (2017) 1–58.
22. R. Metzler, A. Rajyaguru and B. Berkowitz, Modelling anomalous diffusion in semi-infinite disordered systems and porous media, *New J. Phys.* **24** (2022) 123004.
23. H. Sun *et al.*, A new collection of real world applications of fractional calculus in science and

- engineering, *Commun. Nonlinear Sci. Numer. Simul.* **64** (2018) 213–231.
24. R. Metzler, W. G. Glöckle and T. F. Nonnenmacher, Fractional model equation for anomalous diffusion, *Physica A: Statist. Mech. Appl.* **211** (1994) 13–24.
 25. S. A. Silling, Reformulation of elasticity theory for discontinuities and long-range forces, *J. Mech. Phys. Solids* **48** (2000) 175–209.
 26. A. El Abd et al., Determination of moisture distributions in porous building bricks by neutron radiography, *Appl. Radiat. Isotop.* **156** (2020) 108970.
 27. H. Sun et al., A fractal Richards' equation to capture the non-Boltzmann scaling of water transport in unsaturated media, *Adv. Water Resour.* **52** (2013) 292–295.
 28. A. Katiyar et al., A peridynamic formulation of pressure driven convective fluid transport in porous media, *J. Comput. Phys.* **261** (2014) 209–229.
 29. R. Jabakhanji and R. H. Mohtar, A peridynamic model of flow in porous media, *Adv. Water Resour.* **78** (2015) 22–35.
 30. Z. Li et al., Quantifying nonlocal bedload transport: A regional-based nonlocal model for bedload transport from local to global scales, *Adv. Water Resour.* **177** (2023) 104444.
 31. T. Ni et al., Hybrid FEM and peridynamic simulation of hydraulic fracture propagation in saturated porous media, *Comput. Methods Appl. Mech. Eng.* **366** (2020) 113101.
 32. M. Hajipour et al., On an accurate discretization of a variable-order fractional reaction–diffusion equation, *Commun. Nonlinear Sci. Numer. Simul.* **69** (2019) 119–133.
 33. H. Sun et al., Relaxation and diffusion models with non-singular kernels, *Physica A: Statist. Mech. Appl.* **468** (2017) 590–596.
 34. Y. Zhang, D. A. Benson and D. M. Reeves, Time and space nonlocalities underlying fractional-derivative models: Distinction and literature review of field applications, *Adv. Water Resour.* **32** (2009) 561–581.
 35. Q. Du et al., Analysis and approximation of nonlocal diffusion problems with volume constraints, *SIAM Rev.* **54** (2012) 667–696.
 36. S. A. Silling, M. Zimmermann and R. Abeyaratne, Deformation of a peridynamic bar, *J. Elastic.* **73** (2003) 173–190.
 37. H. Li et al., An implicit coupling finite element and peridynamic method for dynamic problems of solid mechanics with crack propagation, *Int. J. Appl. Mech.* **10** (2018) 1850037.
 38. T. Ni et al., Numerical simulation of forerunning fracture in saturated porous solids with hybrid fem/peridynamic model, *Comput. Geotech.* **133** (2021) 104024.
 39. H. Ren et al., Nonlocal operator method for the Cahn–Hilliard phase field model, *Commun. Nonlinear Sci. Numer. Simul.* **96** (2021) 105687.
 40. M. Zaccariotto et al., Examples of applications of the peridynamic theory to the solution of static equilibrium problems, *Aeronaut. J.* **119** (2015) 677–700.
 41. H. Sun, Y. Wang, L. Yu and X. Yu, A discussion on nonlocality: From fractional derivative model to peridynamic model, *Commun. Nonlinear Sci. Numer. Simul.* **114** (2022) 106604.
 42. Z. Chen and F. Bobaru, Selecting the kernel in a peridynamic formulation: A study for transient heat diffusion, *Comput. Phys. Commun.* **197** (2015) 51–60.
 43. Y. Wang et al., A nonlocal fractional peridynamic diffusion model, *Fractal Fract.* **5** (2021) 76.
 44. M. T. Van Genuchten, A closed-form equation for predicting the hydraulic conductivity of unsaturated soils, *Soil Sci. Soc. Amer. J.* **44** (1980) 892–898.
 45. Z. Chen et al., Investigation on influences of the suspension parameters on running stability for suspended monorail vehicle, *J. Railway Sci. Eng.* **15** (2018) 2647–2653.
 46. A. E.-G. El Abd and J. J. Milczarek, Neutron radiography study of water absorption in porous building materials: Anomalous diffusion analysis, *J. Phys. D: Appl. Phys.* **37** (2004) 2305.